CHAPTER 16

THEOREMS RELATED WITH AREA

Animation 16.1: mirandamolina
Source & Credit: The Math Kid
Students Learning Outcomes

After studying this unit, the students will be able to:
• Prove that parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.
• Prove that parallelograms on equal bases and having the same altitude are equal in area.
• Prove that triangles on the same base and of the same altitude are equal in area.
• Prove that triangles on equal bases and of the same altitude are equal in area.

Introduction
In this unit we will state and prove some important theorems related with area of parallelograms and triangles along with corollaries. We shall apply them to solve appropriate problems and to prove some useful results.

Some Preliminaries

Area of a Figure
The region enclosed by the bounding lines of a closed figure is called the area of the figure.
The area of a closed region is expressed in square units (say, sq. m or m²) i.e. a positive real number.

Triangular Region
The interior of a triangle is the part of the plane enclosed by the triangle.
A triangular region is the union of a triangle and its interior i.e., the three line segments forming the triangle and its interior.
By area of a triangle, we mean the area of its triangular region.

Congruent Area Axiom
If \( \triangle ABC \cong \triangle PQR \), then area of (region \( \triangle ABC \)) = area of (region \( \triangle PQR \))

Rectangular Region
The interior of a rectangle is the part of the plane enclosed by the rectangle.
A rectangular region is the union of a rectangle and its interior.
A rectangular region can be divided into two or more than two triangular regions in many ways.
Recall that if the length and width of a rectangle are \( a \) units and \( b \) units respectively, then the area of the rectangle is equal to \( a \times b \) square units.
If \( a \) is the side of a square, its area = \( a^2 \) square units.

Between the same Parallels
Two parallelograms are said to be between the same parallels, when their bases are in the same straight line and their sides opposite to these bases are also in a straight line; as the parallelograms ABCD, EFGH in the given figure.
Two triangles are said to be between the same parallels, when their bases are in the same straight line and the line joining their vertices is parallel to their bases; as the \( \triangle ABC, \triangle DEF \) in the given figure.
A triangle and a parallelogram are said to be between the same parallels, when their bases are in the same straight line, and the side of the parallelogram opposite the base, produced if necessary, passes through the vertex of the triangle as are the \( \triangle ABC \) and the parallelogram DEFG in the given figure.

Definition
If one side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it, is called the Altitude or Height of the parallelogram.
**Definition**

If one side of a triangle is taken as its base, the perpendicular to that side, from the opposite vertex is called the Altitude or Height of the triangle.

**Useful Result**

Triangles or parallelograms placed between the same or equal parallels will have the same or equal altitudes or heights.

Place the triangles ABC, DEF so that their bases BC, EF are in the same straight line and the vertices on the same side of it, and suppose AL, DM are the equal altitudes. We have to show that AD is parallel to BCEF.

**Proof**

\[ m_{AL} = m_{DM} \] (given)

Therefore, AD is parallel to EM.

A similar proof may be given in the case of parallelograms.

**Useful Result**

A diagonal of a parallelogram divides it into two congruent triangles (S.S.S.) and hence of equal area.

**Theorem 16.1.1**

Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.

**Given**

Two parallelograms ABCD and ABEF having the same base AB and between the same parallel lines AB and DE.

**To Prove**

Area of parallelogram ABCD = Area of parallelogram ABEF

**Proof**

\[
\begin{align*}
\text{Area of (parallelogram ABCD)} & = \text{Area of (quad. ABED) + Area of (\triangle CBE)} \ldots (1) \\
\text{Area of (parallelogram ABEF)} & = \text{Area of (quad. ABED) + Area of (\triangle DAF)} \ldots (2)
\end{align*}
\]

In \(\triangle BCF\) and \(\triangle DAF\)

\[
\begin{align*}
m_{\angle CBE} &= m_{\angle DAF} \\
m_{\triangle CBE} &= m_{\triangle DAF} \\
\therefore \triangle CBE & \cong \triangle DAF \\
\therefore \text{Area of (\triangle CBE)} & = \text{Area of (\triangle DAF)} \ldots (3)
\end{align*}
\]

Hence Area of (parallelogram ABCD)

= Area of (parallelogram ABEF)

from (1), (2) and (3)

**Corollary**

(i) The area of a parallelogram is equal to that of a rectangle on the same base and having the same altitude.

(ii) Hence area of parallelogram = base \times\ altitude

**Proof**

Let ABCD be a parallelogram. \(\overline{AL}\) is an altitude corresponding to side \(\overline{AB}\).

(i) Since parallelogram ABCD and rectangle ALMB are on the same base AB and between the same parallels,

\[ \therefore \text{by above theorem it follows that} \]

Area of (parallelogram ABCD) = Area of (rect. ALMB)

(ii) But area of (rect. ALMB) = \(\overline{AB} \times \overline{AL}\)

Hence area of (parallelogram ABCD) = \(\overline{AB} \times \overline{AL}\).
Theorem 16.1.2

Parallelograms on equal bases and having the same (or equal) altitude are equal in area.

Given

Parallelograms ABCD, EFGH are on the equal bases BC, FG, having equal altitudes.

To Prove

area of (parallelogram ABCD) = area of (parallelogram EFGH)

Construction

Place the parallelograms ABCD and EFGH so that their equal bases BC, FG are in the straight line BCFG. Join BE and CH.

Proof

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>The given</td>
<td></td>
</tr>
<tr>
<td>Hence ADEH is a straight line</td>
<td></td>
</tr>
<tr>
<td>mBC = mFG</td>
<td>EFGH is a parallelogram</td>
</tr>
<tr>
<td>mEH = mEH</td>
<td>A quadrilateral with two opposite sides congruent and parallel is a parallelogram</td>
</tr>
<tr>
<td>Now mBC = mEH and they are</td>
<td></td>
</tr>
<tr>
<td>BE and CH are both equal and</td>
<td></td>
</tr>
<tr>
<td>Hence area (</td>
<td></td>
</tr>
<tr>
<td>.......(i)</td>
<td>From (i) and (ii)</td>
</tr>
</tbody>
</table>

EXERCISE 16.1

1. Show that the line segment joining the mid-points of opposite sides of a parallelogram, divides it into two equal parallelograms.
2. In a parallelogram ABCD, mAB = 10 cm. The altitudes corresponding to sides AB and AD are respectively 7 cm and 8 cm. Find AD.
3. If two parallelograms of equal areas have the same or equal bases, their altitudes are equal.

Theorem 16.1.3

Triangles on the same base and of the same (i.e. equal) altitudes are equal in area.

Given

Δ ABC, DBC on the same base BC, and having equal altitudes.

To Prove

area of (ΔABC) = area of (ΔDBC)

Construction

Draw BM || to CA, CN || to BD meeting AD produced in M, N.

Proof

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔABC and ΔDBC are between the same</td>
<td></td>
</tr>
<tr>
<td>Hence MADM is parallel to BC</td>
<td>These</td>
</tr>
<tr>
<td>Area (</td>
<td></td>
</tr>
<tr>
<td>.......(i)</td>
<td>......(ii)</td>
</tr>
<tr>
<td>But Area of ΔABC = ( \frac{1}{2} ) (Area of</td>
<td></td>
</tr>
<tr>
<td>.......(iii)</td>
<td></td>
</tr>
</tbody>
</table>
Theorem 16.1.4

Triangles on equal bases and of equal altitudes are equal in area.

Proof

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\triangle ABC$, $\triangle DEF$ are between the same parallels</td>
<td>Their altitudes are equal (given)</td>
</tr>
<tr>
<td>$\therefore$ XADY is $\parallel$ to BCEF</td>
<td>These $\parallel$ are on equal bases and between the same parallels</td>
</tr>
<tr>
<td>$\therefore$ area ($\parallel gm$ BCAX) = area ($\parallel gm$ EFYD)</td>
<td>Diagonal of a $\parallel gm$ bisects it</td>
</tr>
<tr>
<td>But Area of $\triangle ABC = \frac{1}{2}$ Area of ($\parallel gm$ BCAX)</td>
<td>From (i), (ii) and (iii)</td>
</tr>
</tbody>
</table>

Corollaries

1. Triangles on equal bases and between the same parallels are equal in area.
2. Triangles having a common vertex and equal bases in the same straight line, are equal in area.

EXERCISE 16.2

1. Show that a median of a triangle divides it into two triangles of equal area.
2. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.
3. Divide a triangle into six equal triangular parts.

REVIEW EXERCISE 16

1. Which of the following are true and which are false?
   (i) Area of a figure means region enclosed by bounding lines of closed figure.
   (ii) Similar figures have same area.
   (iii) Congruent figures have same area.
   (iv) A diagonal of a parallelogram divides it into two non-congruent triangles.
   (v) Altitude of a triangle means perpendicular from vertex to the opposite side (base).
   (vi) Area of a parallelogram is equal to the product of base and height.
2. Find the area of the following.
(i) \[ \text{rectangle with sides 6 cm and 3 cm} \]
(ii) \[ \text{square with side 4 cm} \]
(iii) \[ \text{trapezoid with bases 4 cm and 8 cm and height 4 cm} \]
(iv) \[ \text{triangle with base 10 cm and height 16 cm} \]

3. Define the following
(i) Area of a figure
(ii) Triangular Region
(iii) Rectangular Region
(iv) Altitude or Height of a triangle

**SUMMARY**

In this unit we mentioned some necessary preliminaries, stated and proved the following theorems along with corollaries, if any.
- Area of a figure means region enclosed by the boundary lines of a closed figure.
- A triangular region means the union of triangle and its interior.
- By area of triangle means the area of its triangular region.
- Altitude or height of a triangle means perpendicular distance to base from its opposite vertex.
- Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.
- Parallelograms on equal bases and having the same (or equal) altitude are equal in area.
- Triangles on the same base and of the same (i.e. equal) altitudes are equal in area.
- Triangles on equal bases and of equal altitudes are equal in area.