# Unit 7 Permutation and Combination

## INTRODUCTION

In our daily life, permutation and combination play vital role in counting total number of possibilities, in arrangements and selections of objects or things. Permutation and combination are used in many fields of sciences. For example,

- In probability theory, permutation and combination are used to compute how many times an event occurs in various scenarios and used to estimate the odds of winning a lottery.
- In biology, these are used to find out the total numbers of possible DNA sequences.
- In computer science, these are used to count the possible number of passwords of a given length by using some specific characteristics.
- Moreover, these are the important parts of many encryption algorithms to ensure the privacy and integrity of a data set.

**History**

**Augustin Louis Cauchy** (1789 – 1857) is the father of permutation.

**Blaise Pascal and Pierre de Fermat** (1607-1665) gave an idea to generate the combinations of objects.

**Pascal and Leibniz** are the founder of modern combinatorics.

## 7.1 Fundamental Principle of Counting

Danish wants to prepare invitation cards of 5 different colours (red, blue, green, orange and yellow) by changing any of 3 shapes (circle, square and rectangle). How many cards can Danish make?

**Challenge!**
Make a tree diagram and find how many cards can Danish make?

The problem is to count the total number of ways in which Danish can make cards. One way to find the solution is by making tree diagram.

Let us discuss another scenario: Danish's father wants to buy a table and has asked his son to help him decide. He narrowed down his options for manufacturer, types of material (wood, plastic, glass and marble) and types of shape (circle, square and rectangle). Find the total number of table choices from the above options.

Again the problem is to count the total number of ways in which Danish's father can choose a table.

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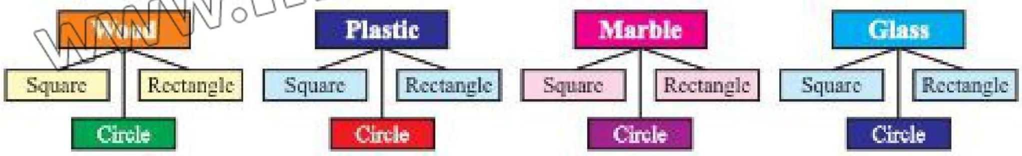
$1^{st}$ Way: By making tree diagram.




From tree diagram, it is clearer there are 12 choices for Danish's father to buy a table with one type of material and one type of shape.

**$2^{nd}$ Way:** By multiplying, Danish's father can find the total number of table choices to buy a table with one kind of material and shape.

Total number of table choices = Total types of material × Total types of shape

$= 4 \times 3 = 12$ choices

These examples show that when making a choice involving multiple stages or categories, we can find the total number of outcomes by multiplying the number of options at each stage.

## Statement

Suppose $A$ and $B$ are two events, the event "$A$" occurs in "$m$" different ways, and the event "$B$" occurs in "$n$" different ways then the total number of ways that the two events together can occur is the product of "$m$" and "$n$".

$$\text{Total number of ways} = mn$$

**Proof:** Let $A = \{a_1, a_2, a_3, \cdots, a_m\}$ and $B = \{b_1, b_2, b_3, \cdots, b_n\}$. Let $P$ denotes the event that both events $A$ and $B$ occur together then $P[(a_i, b_j): a_i \in A,\ b_j \in B,\ 1 \le i \le m,\ 1 \le j \le n\} = A \times B$. Hence the number of ways in which both events $A$ and $B$ can occur is the number of elements in $A \times B$ which is $mn$.

This principle can be extended to three or more events. For instance, if event $A$ can occur in $m$ ways, event $B$ can occur in n ways and event $C$ can occur in $k$ ways, the number of ways that three events can occur all together is the product of $m$, $n$ and $k$.

> **Try yourself**
> If three dice are rolled together, how many total numbers of ways occur?

$$\text{Total number of ways} = m \times n \times k$$

## Factorial (!)

> **History**
> The factorial notation (!) was introduced by Christian Kramp (1760-1826) in 1808
> This notation is frequently used to solve permutation and combination.

Suppose there are four chairs to be occupied by four students and we are interested in counting all the possible ways the students can be seated.

To occupy the first chair there are 4 options. For the second chair, only 3 students remain, so there are 3 options. Similarly, for the third and fourth chairs, there are 2 and 1 options respectively.

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In this way, we have to perform four independent events with 4, 3, 2, and 1 options respectively.

By the **Fundamental Principle of Counting**, the total number of ways to occupy all the chairs is 4.3.2.1 = 24

Such problems frequently occur in daily life, where we to multiply the first $n$ natural numbers: $1, 2, 3, \cdots, n$.

We call this product the factorial of $n$ and denote it by $n!$ Or $\lfloor n$, thus for a natural number $n$:

$n! = \lfloor n = n(n-1)(n-2)\cdots 3.2.1$

For some reason we also define $0! = 1$. In general if $n$ is a non-negative integer, then its factorial is denoted and defined as

$$n! = \lfloor n = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)(n-2)\ldots 3.2.1 & \text{if } n \geq 1 \end{cases}$$

For example,
$1! = 1$
$2! = 2.1 = 2$
$3! = 3.2.1 = 6$
$4! = 4.3.2.1 = 24$
$5! = 5.4.3.2.1 = 120$
$6! = 6.5.4.3.2.1 = 720$

**Challenge!**

Can you find out $\frac{8!}{3!}$?

It can be easily observed that

$n! = n(n-1)!$ for $n \geq 1$

**Example 1:** Evaluate $\frac{8!}{6!}$

**Solution:** $\frac{8!}{6!} = \frac{8.7.6.5.4.3.2.1}{6.5.4.3.2.1} = 56$

**Example 2:** write 8.7.6.5 in the factorial form.

**Solution:** $8.7.6.5 = \frac{8.7.6.5.4.3.2.1}{4.3.2.1} = \frac{8!}{4!}$

**Example 3:** Evaluate $\frac{9!}{6!\,3!}$

**Solution:** $\frac{9!}{6!\,3!} = \frac{(9.8.7)6!}{6!(3.2.1)} = 84$

or $\frac{9!}{6!\,3!} = \frac{9.8.7.6.5.4.3!}{6.5.4.3.2.1.3!} = 84$

or $\frac{9!}{6!\,3!} = \frac{9.8.7.6.5.4.3.2.1}{6.5.4.3.2.1.3.2.1} = 84$

## EXERCISE 7.1

1. Let us make paratha roll. We can choose our fillings from the following:
   **Meat:** Chicken or beef **Vegetable:** Onions, tomatoes or cucumber
   **Sauce:** Mayo or Chutney
   How many different kinds of rolls can we make?

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2. Suppose we have 3 universities, and each offers 4 careers. Use a tree diagram to figure out how many possible career paths you can take.
3. Evaluate each of the following:

(i) $7!$ (ii) $9!$ (iii) $\frac{10!}{8!}$ (iv) $\frac{12!}{9!}$

(v) $\frac{9!}{2!\,7!}$ (vi) $\frac{5!}{2!\,3!\,1!}$ (vii) $\frac{10!}{3!4!}$ (viii) $\frac{12!}{3!3!5!}$

(ix) $\frac{12!}{3!(12-3)!}$ (x) $\frac{20!}{20!(20-20)!}$ (xi) $\frac{8!}{0!}$ (xii) $6!.0!.2!$

4. Write each of the following in the factorial form:

(i) $8.7.6.5$ (ii) $15.14.13.12.11$ (iii) $19.18.17.16$

(iv) $\frac{11.10.9.8.7}{5.4}$ (v) $\frac{10.9.8.7.6}{5.4.3.2.1}$ (vi) $\frac{50.49.48.47}{5.4.3.2.1}$

(vii) $n(n-1)(n-2)(n-3)$ (viii) $(n+2)(n+1)(n)(n-1)$

(ix) $\frac{(n+3)(n+2)(n+1)(n)}{5.4.3.2.1}$ (x) $n(n-1)(n-2)\ldots(n-r+2)$

## 7.2 Permutation

One important application of the fundamental principle of counting is to determine the number of ways that objects can be arranged in order.

**Definition:** An arrangement of all or part of set of objects in a specific order is called a permutation. Number of permutations of $r(\leq n)$ objects taken from a set of $n$ objects is written as ${}^nP_r$ or $P(n, r)$.

$$ {}^nP_r = \frac{n!}{(n-r)!} \quad \text{when } r \leq n $$

According to fundamental principle of counting:

(i) Three books of mathematics for grades 1, 2 and 3 can be arranged in a row taken all at a time (If books are distinct)

$$ {}^nP_r = {}^3P_3 \qquad \because \quad n = r $$

$$ = \frac{3!}{(3-3)!} = \frac{3!}{0!} \qquad \because \quad 0! = 1 $$

$$ = 3! = 3\cdot 2\cdot 1 = 6 \text{ ways} $$

(ii) Number of ways of writing the letters of the WORD taken all at a time

$^nP_r = {^4P_4}$ $\quad\because\ n = r$

$= \dfrac{4!}{(4-4)!} = \dfrac{4!}{0!}$ $\quad\because\ 0! = 1$

$= 4! = 4.3.2.1 = 24$ ways

n = Total number of things/objects
r = The number of selected things / objects

**Challenge!**

Can you make total number of permutations for the "WORD" pictorially?

**Do you know!**

In 1974, "Erno Rubik" invented a popular puzzle, each turn of the puzzle shows a permutation of the different colours. The name of this puzzle is "Rubik's Cube".

**Theorem:** Prove that: $^nP_r = n(n-1)(n-2)....(n-r+1) = \dfrac{n!}{(n-r)!}$

**Proof:** As there are $n$ different objects to fill up $r$ places. So, the first place can be filled in $n$ ways. Since repetitions are not allowed, so after placing one object we are left with $(n-1)$ objects, thus the second place can be filled in $(n-1)$ ways. Similarly the third place can be filled in $(n-2)$ ways, and so on. This continues until the $r^{th}$ place which can be filled in $n-(r-1) = n-r+1$ ways. Therefore, by the **Fundamental Principle of Counting**, $r$ places can be filled by $n$ different objects in $n(n-1)(n-2)....(n-r+1)$ ways.

$$^nP_r = n(n-1)(n-2)...(n-r+1)$$

$$= \frac{n(n-1)(n-2)...(n-r+1)(n-r)!}{(n-r)!}$$

$$^nP_r = \frac{n!}{(n-r)!}$$

**Example 4:** How many different 4-digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 6, when no digit is repeated?

**Solution:** The total number of digits = 6

The digits forming each number = 4.

So, the required number of 4-digit numbers is given by:

$$^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6.5.4.3.2.1}{2.1} = 6.5.4.3 = 360$$

**Example 5:** In how many ways can a set of 4 different mathematics books, 3 different physics books and 2 different chemistry books be placed on a shelf with a space for 9 books, if:

(a) All the books are kept without any restriction.

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(b) All the books of the same subject are kept together.
(c) Only the mathematics books are kept together.

**Solution:**

(a) All the books are kept without any restriction.

Total number of books $= 4 + 3 + 2 = 9$


$^9P_9 = 9! = 9.8.7.6.5.4.3.2.1$

$= 362880$ ways

**Reason for defining 0! = 1**

If $n = r$, then $^nP_r = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$

$\boxed{n(n-1)\ \ldots\ 3.2.1 = n! = \dfrac{n!}{0!}}$ $\boxed{\Rightarrow\ 0! = 1}$

(b) All the books of the same subject are kept together.

$^4P_4 \,.\, ^3P_3 \,.\, ^2P_2 \,.\, ^3P_3 = 4!\,.\,3!\,.\,2!\,.\,3!$

$= 3!\,.\,24\,.\,6\,.\,2\,.\,6$

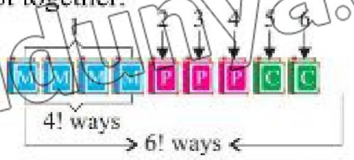
$= 1728$ ways


(c) Only the mathematics books are kept together.

$^4P_4 \,.\, ^6P_6 = 4!\,.\,6!$

$= 24\,.\,720$

$= 17280$ ways




**Example 6:** In how many ways 5 people are to be seated on a bench if:

(a) there are no restrictions
(b) two people can sit next to each other
(c) two people cannot sit next to each other.

**Challenge!**

Find the number of ways if only physics books are kept together.

**Solution:**

(a) when there is no restriction, then

Number of ways $= {}^5P_5 = 5! = 120$


(b) when two people can sit next to each other, then

$= {}^4P_4 \,.\, {}^2P_2$

$= 4! \cdot 2! = 24 \cdot 2$

$= 48$ ways

A and B is considered as 1 unit.


(c) when two people cannot sit next to each other, then

$= {}^5P_5 - [2$ can sit next to each other$]$

$= 5! - 48 = 120 - 48$

$= 72$ way

**Try yourself**

In how many ways 6 people are to seated on a table if 3 cannot sit next to each other?

# EXERCISE 7.2

1. Evaluate the following:

   (i) $^{10}P_5$ (ii) $^{5}P_2$ (iii) $^{7}P_7$ (iv) $^{10}P_3$

2. Find the value of $n$ when:

   (i) $^{n}P_3 = 504$, (ii) $^{15}P_n = 15.14.13.12\ 11$ (iii) $^{n}P_5 : {}^{n-2}P_2 = 540 : 1$

3. Prove from the first principle that:

   (i) $^{n}P_r = n\ .\ {}^{n-1}P_{r-1}$ (ii) $^{n}P_r = {}^{n-1}P_r + r\ .\ {}^{n-1}P_{r-1}$

4. How many signals can be given by 6 flags of different colours, using 2 flags at a time?

5. From a deck of 13 cards, find in how many ways these are arranging in a rectangular form? Hint (order is matter)

   (i) All cards (ii) 8 cards (iii) 10 cards

6. In how many ways can the seven alphabet a e f i o u h be arranged in a row?

7. There are 8 men. Find the number of ways of arranging them in a row if:

   (a) Two old men are at left side

   (b) The youngest man is not at the right side

8. How many arrangements are there, if 6 books are arranged in a row out of 12 books?

9. Find permutation of 10 people sitting on a bench if:

   (a) There are no restriction (b) 3 cannot sit next to each other.

10. In how many ways can a set of 4 different blue pens, 3 different red pens and 6 black pens be placed in a rectangular form rack with a space for 10 pens if:

    (a) All the pens are placed without any restriction

    (b) All the pens of the same colour are placed together

    (c) Only the red pens are placed together

11. Hamza wants to distribute 15 pencils among 6 needy children in this way that the youngest gets 4 pencils and others get 2 pencils. Find how many ways, there are of arranging in a row form?

12. In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together?

13. Find the number of arrangements of 3 books on English and 5 books on Urdu for placing them on a shelf such that the books on the same subject are together.

14. In how many ways can 5 boys and 4 girls be seated on a bench so that the girls and the boys occupy alternate seats?

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## 7.3 Permutation of Objects Not All Different

Suppose we have to find the permutations of the letters of the word BITTER using all the letters in it. The word BIT$_1$T$_2$ER consists of 6 different letters which can be permuted among themselves in 6! ways.

We can see that all the letters of the word BITTER are not different. It has 2Ts in it. After replacing 2Ts, we can see there are 2! ways.

B I T$_1$T$_2$E R  B I T$_2$T$_1$E R

2! ways

The replacement of the two Ts by $T_1$ and $T_2$ in any other permutation will give rise to 2 permutations.

Hence, the number of permutations of the letters of the word BITTER taken all at a time.

$$\frac{6!}{2!}=\frac{6.5.4.3.2.1}{2.1}=360 \text{ ways}$$

**Remember!**

If there is $n_1$ alike objects of one kind, $n_2$ alike objects of second kind and $n_3$ alike objects of third kind, then the number of permutations of $n$ objects taken all at a time is given by:

$$\frac{n!}{n_1!\cdot n_2!\cdot n_3!}=\binom{n}{n_1,\ n_2,\ n_3}$$

**Example 7:** In how many ways can be letters of the word $\boxed{\text{MISSISSIPPI}}$ be arranged when all the letters are to be used?

**Solution:** Total number of letters in the word =11

$\boxed{\text{MISSISSIPPI}}$

I is repeated 4 times = 4! ways

S is repeated 4 times = 4! ways

P is repeated 2 times = 2! ways

M comes once only = 1! ways

$$\text{Required number of permutations} = \frac{11!}{4!\cdot 4!\cdot 2!\cdot 1!}=34650 \text{ ways}$$

### Circular Permutation

The permutation in which the objects are arranged in a circular order is known as circular permutation.

**Note:**

The following circular arrangements are reflection of each other and considered same when anticlockwise and clockwise arrangements are considered identical.


Circular permutation can occur in two cases:

**Case-I: When clockwise and anticlockwise arrangements are considered different**

In a linear arrangement, changing the order of objects results in a new arrangement. However, in a circular arrangement, rotating the entire circle does not produce a new, distinct arrangement.

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For example, suppose three people A, B, and C are sitting around a round table. The following three linear arrangements

A – B – C, B – C – A and C – A – B are all considered the same in circular permutations because each one is simply a rotation of the others.

We conclude that:

3 linear permutations gives 1 circular permutation.

3! linear permutations gives $\frac{1}{3}.3! = \frac{3!}{3} = 2!$ permutations.

Generalizing the above idea if $n$ objects are arranged in a circle, the number of distinct circular permutations is $\frac{n!}{n} = (n-1)!$

**Case-II: When clockwise and anticlockwise arrangements are considered identical**

In many real-life situations, a circular permutation and its mirror image are not considered different.

For example, if three beads red, blue, and black are arranged in a necklace, then an arrangement and its reflection (as shown in the figure) are considered the same.

In such cases, we divide the total number of circular permutations by 2 to eliminate symmetrical duplicates.

Thus, the number of distinct circular permutations is:

$$\frac{(n-1)!}{2}$$

**Example 8:** In how many ways can 4 persons be seated at a round table, while:

(i) clockwise and anticlockwise orders are different

(ii) clockwise and anticlockwise orders are identical.

**Solution:** Let $A$, $B$, $C$ and $D$ be the 4 persons.

(i) If clockwise and anticlockwise orders are different

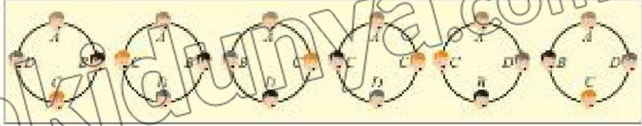
**According to Case-I**

The possible number of ways are:

$= (n-1)!$ ways

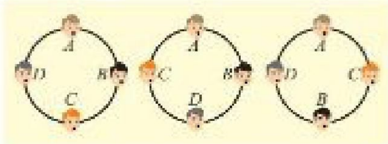
$= (4-1)! = 3!$

$= 3.2.1 = 6$ ways.



(ii) If clockwise and anticlockwise orders are identical

**According to Case-II**




The possible number of ways are $= \frac{(n-1)}{2}$ ways

$$= \frac{(4-1)!}{2} = \frac{3!}{2}$$

$$= \frac{3\cdot 2}{2} = 3 \text{ ways}$$

## EXERCISE 7.3

1. How many arrangements of the letters of the following words, taken all together can be made?

   (i) CURRICULUM (ii) ADSORPTIVELY (iii) PROBABILITY

2. A girl has 9 marbles. There are 4 red marbles, 3 blue, and 2 green marbles. If she arranges them in a row, then find in how many different arrangements she can make take all at time?

3. In how many different ways can the following persons sit in a round table?

   **Hint** (Solve according to both the cases)

   (a) 8 persons (b) 7 persons (c) 6 persons

4. In how many ways can 5 couples sit on a round table if no two women are sitting together?

5. How many arrangements of the letters of the word ATTACKED can be made if each arrangement begins with C and ends with K?

6. How many 6-digit numbers can be formed from the digits 7, 7, 8, 8, 9, 9?

7. 15 members of a club form 4 committees of 3, 5, 4, 3 members so that no member is a member of more than one committee. Find the number of committees.

8. The D.C.Os of 11 districts meet to discuss the law-and-order situation in their districts. In how many ways can they be seated at a round table, when two particular D.C.Os insist on sitting together?

9. The Governor of the Punjab calls a meeting of 14 officers. In how many ways can they be seated at a round table?

10. Fatima invites 14 people to a dinner. There are 9 males and 5 females who are seated at two different tables. Guests of one sex sit at one round table and the guests of the other sex sit at the second table. Find the number of ways in which all guests are seated.

11. Find the number of ways in which 5 men and 5 women can be seated at a round table in such a way that no two persons of the same sex sit together.
12. In how many ways can 6 keys be arranged in a circular key ring?
13. How many necklaces can be made from 6 beads of different colours?

## 7.4 Combination

Suppose, a teacher uses the names of few students to make a team for a writing competition. Such as Ahmad, Sana, Hamza and Danish. As a combination of team members, (Ahmad, Sana, Hamza and Danish) is equivalent to (Hamza, Ahmad, Danish and Sana ). Because same students are in the combination. Consequently, you have the same team because the order of the name of the students does not matter.

| Ahmad | Sana | Hamza | Danish |
|---|---|---|---|
| Hamza | Ahmad | Danish | Sana |

So, we are interested in the membership of the team and not in the ways the students are listed (arranged).

### Definition

A combination of $r$ objects taken out of $n$ objects is a subset of $r$ objects of a set of $n$ objects.

The number of combinations of $n$ different objects taken $r$ at a time is denoted by $^nC_r$ or $C(n, r)$ or $\binom{n}{r}$ and is given by $^nC_r = \dfrac{n!}{r!(n-r)!}$.

**Theorem:** Prove that $^nC_r = \dfrac{n!}{r!(n-r)!}$.

**Proof:** Elements of a subset of $r$ objects of a set of $n$ objects can be arranged among themselves in $r!$ ways. So, each combination will give rise to $r!$ permutation. Thus, there will be $^nC_r \times r!$ permutations of $n$ different objects taken $r$ at a time that is:

$$^nC_r \times r! = {}^nP_r$$

$$\Rightarrow \quad {}^nC_r \times r! = \frac{n!}{(n-r)!} \qquad \therefore \quad {}^nC_r = \frac{n!}{r!(n-r)!}$$

Which completes the proof.

Need to know

**Need to know!**

The formulae $^nP_r$ and $^nC_r$ are also known as counting formulae. Because, they are used to count the possible number of ways without listing them all.

### Corollary:

(i) If $r = n$, then $^nC_n = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!\,0!} = 1$

(ii) If $r = 0$, then $^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{0!\,n!} = 1$

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## 7.4.1 Applications of Combination in Real Life

**Example 9:** Zain has 8 different fruits. He wants to select 5 fruits out of 8 fruits to make a fruit chart. How many combinations of fruits he can select?

**Solution:** To solve this problem, we have to find the number of combinations of 5 fruits out of 8 fruits. In this situation, $n = 8$ and $r = 5$.

$$^nC_r = \frac{n!}{r!(n-r)!}$$

After putting values

$$^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\,.\,3!}$$

$$= \frac{8\times 7\times 6\times 5!}{5!\,.\,3!} = \frac{8\times 7\times \cancel{6}}{\cancel{3.2}\,.1}$$

$$= 8\times 7 = 56 \text{ ways}$$

Zain has 56 different ways to select 5 different fruits to make a fruit chart.

**Example 10:** In a school, a class consists of 12 girls and 8 boys. The teacher wants to select 5 students for an activity. In how many ways can the students be selected including? (i) 2 girls (ii) 5 boys (iii) 2 boys

**Solution:** Number of girls = 12

Number of boys = 8

**Challenge!**

A restaurant offers 6 flavours of pizza. How many ways are there to select 2 flavours of pizza?

(i) Now let's find the total number of ways to select students when exactly 2 are girls.

$$\left({}^{12}C_2\right)\left({}^{8}C_3\right) = \frac{12!}{2!\,10!}\cdot\frac{8!}{3!\,5!} = \frac{12\cdot 11\cdot 10!}{2\cdot 10!}\cdot\frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 1\cdot 5!} = 3696$$

(ii) Let's find total number of ways to select students when exactly 5 students are boys.

$$^8C_5 = \frac{8!}{5!\,(8-5)!} = \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot 6\cdot 5!}{5!\,3\cdot 2\cdot 1} = 56$$

(iii) Let's find total number of ways to select students when exactly 2 students are boys.

$$\left({}^{8}C_2\right)\left({}^{12}C_3\right) = \frac{8!}{2!\,6!}\cdot\frac{12!}{3!\,9!} = \frac{8\cdot 7\cdot 6!}{2\cdot 6!}\cdot\frac{12\cdot 11\cdot 10\cdot 9!}{3\cdot 2\cdot 1\cdot 9!} = 36960$$

## 7.4.2 Complementary Combinations

**Theorem.** Prove that: $^nC_r = {}^nC_{n-r}$

**Proof:** If from $n$ different objects, we select $r$ objects then $(n - r)$ objects are left. Corresponding to every combination of $r$ objects, there is a combination of $(n - r)$

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objects and vice versa. Thus, the number of combinations of $n$ objects taken $r$ at a time is equal to the number of combinations of $n$ objects taken $(n-r)$ at a time.

$$\therefore \quad {}^nC_r = {}^nC_{n-r}$$

$${}^nC_{n-r} = \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

$${}^nC_{n-r} = {}^nC_r$$

**Note:**

This result will be found useful in evaluating ${}^nC_r$ when $r > \frac{n}{2}$.

For example,

$${}^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}C_2 = \frac{(12).(11)}{2} = 6.11 = 66$$

**Example 11:** Find the number of the diagonals of a 6-sided figure.

**Solution:** A 6-sided figure has 6 vertices by joining any two vertices, we get a line segment.

$$\therefore \quad \text{Number of line segments} = {}^6C_2 = \frac{6!}{2!\,4!} = 15$$

But these line segments include 6 sides of the figure

$$\therefore \quad \text{number of diagonals} = 15 - 6 = 9$$

### Difference between permutation and combination

| Permutation | Combination |
|---|---|
| • Order is important.<br>e.g., *ab* and *ba* are different (because order of any object is matter)<br>• Arrangement of objects<br>e.g. arrangement of:<br>* ball of different colours<br>* English alphabet (letters)<br>* people while sitting on chairs | • Order is not important<br>e.g., *ab* and *ba* are same (because order does not matter)<br>• Selection of objects<br>e.g. selection of:<br>* different colours<br>* members in a team<br>* food items |

### Application of Permutations and Combinations in Cryptography

**Example 12:** Zain wants to generate a password for his laptop to secure the data. He can take only 6 characters to generate a password. Each character can either be an upper case letter $(A - Z)$ or digits from $(0 - 9)$.

Can you tell how many passwords can be generated by using the above letters and digits:

(i) If repetition of characters is not allowed

(ii) If repetition of characters is allowed

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**Solution:** Total number of letters = 26
Total number of digits = 10
Total number of letters and digits = 26 + 10 = 36
$n$ = total number of characters = 36
$r$ = required number of characters = 6

(i) If repetition of characters is not allowed, we find out total possible permutations as.

$$^{n}P_{r} = {}^{36}P_{6} = \frac{36!}{(36-6)!} = \frac{36!}{30!}$$
$$= \frac{36\cdot 35\cdot 34\cdot 33\cdot 32\cdot 31\cdot 30!}{30!}$$
$$= 36\cdot 35\cdot 34\cdot 33\cdot 32\cdot 31$$
$$= 1,402,410,240 \quad \text{ways}$$

Hence, $1,402,410,240$ passwords can be generated by using the 26 alphabet and 10 digits. (If repetition of the characters is not allowed)

(ii) If the repetition of the characters is allowed. Using Fundamental Principle of Counting:

The total number of possible combinations = 36 × 36 × 36 × 36 × 36 × 36 = $36^6$

Hence, $36^6$ passwords can be generated by using the 26 alphabets and 10 digits, If repetition of characters is not allowed.

**Application of permutations to estimate the odd of winning the lottery.**

**Example 13:** A box contains 15 cards from (1 – 15). Danish is to select 5 cards. If all the selected cards are the first five multiples of 2 then Danish will win the game. Find Danish's chance of winning the game, when

(i) Order is important (ii) Order is not important

**Solution:** $n$ = total number of cards = 15
$r$ = required number of cards = 5

(i) When order is important,

$$\text{Total possible ways} = {}^{n}P_{r} = {}^{15}P_{5} = \frac{15!}{(15-5)!}$$
$$= \frac{15!}{10!} = 360,360 \text{ ways}$$

Hence, Danish's chance to win the game = $\frac{1}{360,360} = 0.000002775$

(ii) When order is not important

$n$ = Total number of cards = 15

$r$ = Required number of cards = 5

$$\text{Total possible ways} = {}^{n}C_r = {}^{15}C_5 = \frac{15!}{5!(15-5)!}$$

$$= \frac{15!}{5!\,10!} = \frac{15\times 14\times 13\times 12\times 11\times \cancel{10!}}{5!\,.\,\cancel{10!}}$$

$$= \frac{15\times 14\times 13\times 12\times 11}{5\times 4\times 3\times 2\times 1} = 3003 \text{ ways}$$

$$\text{Hence, Danish's chance to win the game} = \frac{1}{3003} = 0.00033$$

**Application of Permutation and Combination to choose different sets of songs for Certain Occasions**

**Example 14:** On Independence Day, a DJ has a list of ten different national songs. He wants to select any five national songs for the day. Find how many ways he can select and play the songs:

(i) If the order of playing the songs matters

(ii) If the order of playing the songs does not matter.

**Solution:** (i) When order matters

$n$ = total number of national songs = 10

$r$ = required number of national songs = 5

$$\text{Total number of ways} = {}^{n}P_r = {}^{10}P_5$$

$$= \frac{10!}{(10-5)!} = \frac{10!}{5!} = 30,240 \text{ ways}$$

Hence, the DJ can play the five national songs in 30,240 different ways.

(ii) When order is not matter

$n$ = total number of national songs = 10

$r$ = total number of selected national songs = 5

$$\text{Total number of ways} = {}^{n}C_r = {}^{10}C_5 = \frac{10!}{5!(10-5)!}$$

$$= \frac{10!}{5!\,.\,5!} = 252 \text{ ways}$$

Hence, the DJ can play the five national songs in 252 different ways.

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# EXERCISE 7.4

1. Evaluate the following:

   (i) ${}^5C_3$ (ii) ${}^8C_5$ (iii) ${}^3C_3$ (iv) ${}^{10}C_7$

2. Find the value of $n$, when

   (i) ${}^nC_6 = {}^nC_2$ (ii) ${}^nC_{11} = \dfrac{14.13.12}{3!}$ (iii) ${}^nC_5 = {}^nC_{10}$

3. In how many ways can five subjects be selected out of eight subjects to select a course programme?

4. Find how many ways there are to choose vowel words from the letter of English alphabet?

5. In how many ways 3 dishes of Desi foods and 2 dishes of Chinese foods be selected from 6 dishes of desi foods and 8 dishes of Chinese foods?

6. From a standard deck of 52 playing cards, there are 26 black cards and 26 red cards. How many different possible ways are made of eight cards if select 3 cards of black colour and others are of red colour?

7. A bag contains 8 red balls, 7 green balls. Find the total number of possible ways in which five balls are selected in a way:

   (i) 3 red and 2 green (ii) 1 red and 4 green
   (iii) 4 red and 1 green (iv) All the red balls

8. How many (a) diagonals and (b) triangles can be formed by joining the vertices of the polygon having:

   (i) 5 sides (ii) 8 sides (iii) 12 sides?

9. The members of a club are 10 boys and 8 girls. In how many ways can a committee of 6 boys and 3 girls be formed?

10. How many committees of 5 members can be chosen from a group of 8 persons when each committee must include 2 particular persons?

11. In how many ways can a hockey team of 11 players be selected out of 15 players? how many of them will include a particular player?

12. Show that: ${}^{20}C_7 + {}^{20}C_6 = {}^{21}C_7$

13. There are 6 men and 8 women members of a club. how many committees of seven can be formed?

    (i) 3 women (ii) at the most 3 women (iii) at least 5 women?

14. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

15. A locker of a bank is locked with four letters (A-Z). How many different passwords can be generated if:
    (a) repetition of the alphabets is allowed
    (b) repetition of the alphabets is not allowed
16. Using a cryptographic system, a password is generated with 8 characters. Each character can either be a lowercase letter (a–f) or a digit (0–5). How many passwords can be generated if each password must contain exactly 5 lowercase letters and 3 digits?
    (a) With repetition allowed (b) Without repetition.
17. An urn contains the first 15 English letters (A–O). Sania is to randomly select 3 letters from the urn. She will win the game if the selected letters are the first three vowel letters. Find the probability of Sania winning the game if:
    (a) The order of the vowel letters matters
    (b) The order of the vowel letters does not matter
18. On Defense Day, Teacher I prepares a list of 10 national songs, and Teacher II also prepares a separate list of 10 different national songs. The principal wants to select 3 songs from Teacher I's list and 3 songs from Teacher II's list. In how many ways can the songs be selected if:
    (i) The sequence of the selected songs matters
    (ii) The sequence of the selected songs does not matter.

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