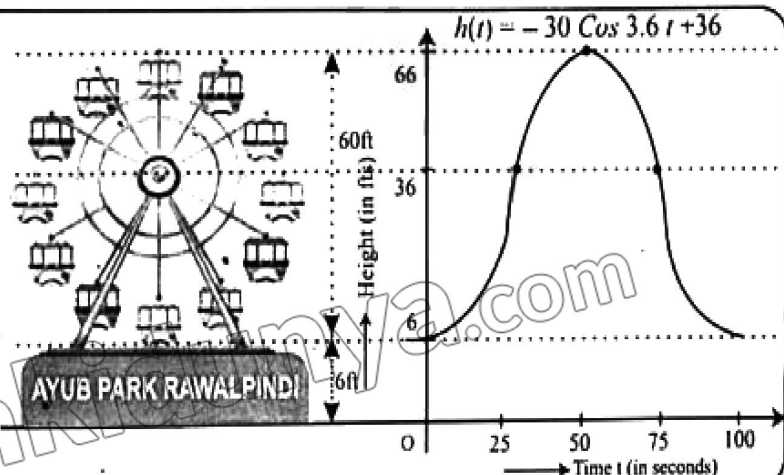


TRIGONOMETRIC FUNCTIONS

After studying this unit, students will be able to:

- Find the domain and range of the trigonometric functions.
- Discuss even and odd functions, and the periodicity of trigonometric functions.
- Find the maximum and minimum value of a given function of the type:
 - $a + b \sin \theta$,
 - $a + b \cos \theta$,
 - $a + b \sin (c\theta + d)$,
 - $a + b \cos (c\theta + d)$,
 - The reciprocal of above, where a , b , c and d are real numbers.
- Graph and analyze the trigonometric functions sine, cosine, and tangent to solve problems.
- Explain the properties of graphs of $\sin \theta$, $\cos \theta$ and $\tan \theta$.
- Apply the concepts of trigonometric functions, identities, graphs, periodicity, even, odd functions, and extreme values to real-world problems such as (distance, elevation, and direction of tall structures, navigation and mapping, lengths of irregular shapes, graphs to visualize and predict patterns in data, frequency and periodic length of Ferris wheel, forces on a see-saw or lever, the ideal angle for solar panel placement)

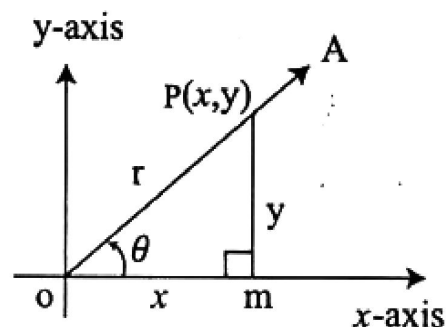
A Ferris wheel is 60 ft. in diameter. It makes one revolution every 100 seconds. Yasir climb up 6 feet of stairs to get on the wheel at its lowest point. Model the height of a rider as a sinusoidal function and graph 1 revolution.



9.1 Domain and Range of Trigonometric Functions

The domain of a function $f(x)$ is the set of all possible values of ' x ' such that function $f(x)$ is defined. The range of a function $f(x)$ is the set of all possible values the function $f(x)$ can take, when ' x ' is any number from the domain of the function.

Let ' θ ' be any real number. Construct the angle whose measure is θ radian, with vertex at the origin of a rectangular coordinate system. Let the initial line of the angle ' θ ' be along x -axis. Let $P(x, y)$ be any point on the terminal side OA of the angle.



Let $\overline{OP} = r$

Then from r.a.t OMP:

$$\begin{aligned} \text{(i)} \quad \sin \theta &= \frac{y}{r} \\ \text{For any } \theta, \quad -r &\leq y \leq r \\ \therefore \quad -1 &\leq \frac{y}{r} \leq 1 \\ &\therefore -1 \leq \sin \theta \leq 1 \end{aligned}$$

$\therefore$ Domain of $\sin \theta = \mathbb{R}$ or $(-\infty, \infty)$ and range of $\sin \theta = [-1, 1]$

(ii) Again, from r.a.t OMP:

$$\begin{aligned} \cos \theta &= \frac{x}{r} \\ \text{For any } \theta, \quad -r &\leq x \leq r \\ \therefore \quad -1 &\leq \frac{x}{r} \leq 1 \\ &\therefore -1 \leq \cos \theta \leq 1 \end{aligned}$$

$\therefore$ Domain of $\cos \theta = \mathbb{R}$ or $(-\infty, \infty)$ and range of $\cos \theta = [-1, 1]$

(iii) Since $\tan \theta$ is defined for any real number ' θ ' which is not an odd multiple of $\frac{\pi}{2}$.

For any such value of ' θ ', the ratio $\frac{y}{x}$ can be any real number.

$\therefore$ Domain of $\tan \theta = \mathbb{R} - \{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$

Range of $\tan \theta = \mathbb{R}$

- Key Facts**

 - $[a, b]$ is called closed interval of numbers from a to b and $[a, b] = a \leq x \leq b$.
 - (a, b) is open interval such that $(a, b) = a < x < b$.

(iv) Cot θ is defined for any real number ' θ ' which is not an even multiple of $\frac{\pi}{2}$.

For any such value of ' θ ', the ratio $\frac{x}{y}$ can be any real number.

$$\therefore \text{Domain of Cot } \theta = \mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$$

$$\text{Range of Cot } \theta = \mathbb{R}$$

(v) Sec θ is defined for any real number ' θ ' which is not an odd multiple of $\frac{\pi}{2}$.

For any such value of ' θ ', we have:

$$\therefore -1 \leq \frac{x}{r} \leq 1 \quad \text{i.e., } \left| \frac{x}{r} \right| \leq 1$$

$$\Rightarrow \left| \frac{1}{\frac{x}{r}} \right| \geq 1 \Rightarrow \left| \frac{r}{x} \right| \geq 1 \Rightarrow |\sec \theta| \geq 1$$

$$\therefore \text{Domain of Sec } \theta = \mathbb{R} - \{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$$

$$\text{Range of Sec } \theta = \mathbb{R} - (-1, 1)$$

(vi) Cosec θ is defined for any real number ' θ ' which is not an even multiple of $\frac{\pi}{2}$.

For any such value of ' θ ', we have:

$$\therefore -1 \leq \frac{y}{r} \leq 1 \quad \text{i.e., } \left| \frac{y}{r} \right| \leq 1$$

$$\Rightarrow \left| \frac{1}{\frac{y}{r}} \right| \geq 1 \Rightarrow \left| \frac{r}{y} \right| \geq 1 \Rightarrow |\csc \theta| \geq 1$$

$$\therefore \text{Domain of Cosec } \theta = \mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$$

$$\text{Range of Cosec } \theta = \mathbb{R} - (-1, 1)$$

Remark-I

The maximum (i.e., greatest) value of Sin θ and Cos θ is 1 and minimum (i.e., least) value is -1.

$$\therefore \quad (i) \quad |\sin \theta| \leq 1; \text{ i.e., } -1 \leq \sin \theta \leq 1; \text{ i.e., } \sin^2 \theta \leq 1.$$

$$(ii) \quad |\cos \theta| \leq 1; \text{ i.e., } -1 \leq \cos \theta \leq 1; \text{ i.e., } \cos^2 \theta \leq 1.$$

Remark-II

Tan θ and Cot θ can take any real number value.

Remark-III

Sec θ and Cosec θ cannot take value in the interval $(-1, 1)$.

$$\therefore |\sec \theta| \geq 1, \text{ i.e., } \sec^2 \theta \geq 1$$

$$|\csc \theta| \geq 1, \text{ i.e., } \csc^2 \theta \geq 1$$

Trig. Functions	Domain	Range
$y = \sin x$	$R = (-\infty, \infty)$	$-1 \leq y \leq 1$
$y = \cos x$	$R = (-\infty, \infty)$	$-1 \leq y \leq 1$
$y = \tan x$	$\{x : x \in R \text{ and } x \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\}$	$R = (-\infty, \infty)$
$y = \cot x$	$\{x : x \in R \text{ and } x \neq n\pi, n \text{ an integer}\}$	$R = (-\infty, \infty)$
$y = \sec x$	$\{x : x \in R \text{ and } x \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\}$	$1 \leq y \text{ and } y \leq -1$
$y = \csc x$	$\{x : x \in R \text{ and } x \neq n\pi, n \text{ an integer}\}$	$1 \leq y \text{ and } y \leq -1$

Example:

Find domain and range of the following:

(i) $y = 4 \sin 3x$

(ii) $y = \frac{1}{2 \cos x - 1}$

(iii) $y = \frac{1}{1 + 2 \sin \theta}$

(iv) $y = \frac{1}{2 - \sin 3x}$

Solution: (i)

We are given

$$y = 4 \sin 3x \dots \dots (i)$$

Since (i) is defined for all real values of 'x'. Then, domain of $y = D_y = (-\infty, \infty)$ or R .

Similarly, to find the range of (i). Let,

$$\Rightarrow 3x = \theta$$

Form (i) $y = 4 \sin \theta \dots \dots (ii)$

As, range of Sine function is $-1 \leq \sin \theta \leq 1$.

So, $-1 \leq \sin \theta \leq 1$

and $-4 \leq 4 \sin \theta \leq 4$

$$-4 \leq 4 \sin 3x \leq 4$$

Form (ii) $-4 \leq y \leq 4$, thus

$$\text{Range of } y = R_y = [-4, 4].$$

(ii)

We are given

$$y = \frac{1}{2 \cos x - 1} \dots \dots (ii)$$

Since (i) is defined for all real values of 'x' except

$$\{x : x \in R \text{ and } x \neq (2n\pi + \frac{\pi}{3}) \wedge x \neq (2n\pi + \frac{5\pi}{3}), n \in Z\}.$$

Therefore, domain of $y = D_y = R - \{x : x \in R \text{ and } x \neq (2n\pi + \frac{\pi}{3}) \wedge x \neq (2n\pi + \frac{5\pi}{3}), n \in Z\}$.

Now, we find the range of (i), since, range of cosine function is $[-1, 1]$.

We have

$$-1 \leq \cos x \leq 1$$

$$-2 \leq 2 \cos x \leq 2$$

$$-2 - 1 \leq 2 \cos x - 1 \leq 2 - 1$$

$$-3 \leq 2 \cos x - 1 \leq 1$$

We get

$$1 \leq \frac{1}{2 \cos x - 1} \leq \frac{-1}{3}$$

From (ii)

$$1 \leq y \leq \frac{-1}{3}$$

$$y \leq \frac{-1}{3} \text{ and } 1 \leq y$$

Thus,

$$\text{Range of } y = R_y = (-\infty, -\frac{1}{3}] \cup [1, +\infty)$$

(iii)

We are given

$$y = \frac{1}{1 + 2 \sin x} \dots \dots \dots (iii)$$

Since (i) is defined for all real values of 'x' except

$$\{x : x \in R \text{ and } x \neq (2n\pi + \frac{7\pi}{6}) \wedge x \neq (2n\pi + \frac{11\pi}{6}), n \in Z\}$$

Therefore, domain of $y = D_y = R - \{x : x \in R \text{ and } x \neq (2n\pi + \frac{7\pi}{6}) \wedge x \neq (2n\pi + \frac{11\pi}{6}), n \in Z\}$.

Now, we find the range of (i), since, range of sine function is $[-1, 1]$.

Similarly,

$$-1 \leq \sin x \leq 1$$

$$-2 \leq 2 \sin x \leq 2$$

$$-2 + 1 \leq 2 \sin x + 1 \leq 2 + 1$$

$$-1 \leq 2 \sin x + 1 \leq 3$$

We get

$$\frac{1}{3} \leq \frac{1}{2 \sin x + 1} \leq -1$$

From (iii)

$$\frac{1}{3} \leq y \leq -1$$

$$\frac{1}{3} \leq y \text{ and } y \leq -1$$

Therefore,

$$\text{Range of } y = R_y = (-\infty, -1] \cup [\frac{1}{3}, +\infty)$$

(iv)

We are given

$$y = \frac{1}{2 - \sin 3x} \quad \dots \dots \dots (iv)$$

To find the domain of (i), we can see that $2 - \sin 3x$ should not be equal to "0", i.e.,

$$2 - \sin 3x \neq 0$$

$$\sin 3x \neq 2$$

Which is understood because $-1 \leq \sin 3x \leq 1$, so (i) is defined for all real values of 'x'.

Hence,

$$\text{Domain of } y = D_y = (-\infty, \infty) \text{ or } \mathbb{R}.$$

Now, we find the range of (i), since, range of sine function is $[-1, 1]$.

We have

$$-1 \leq \sin 3x \leq 1$$

$$-(-1) \geq -\sin 3x \geq -1$$

$$1 \geq -\sin 3x \geq -1$$

$$2 + 1 \geq 2 - \sin 3x \geq 2 - 1$$

$$3 \geq 2 - \sin 3x \geq 1$$

From (iv)

$$\frac{1}{3} \leq \frac{1}{2 - \sin 3x} \leq 1$$

Therefore,

$$\text{Range of } y = R_y = \left[\frac{1}{3}, 1\right]$$

9.2 Periodicity of Trigonometric Functions

We often encounter periodic phenomenon in the nature, technology, and human society. Recall the 24-hour day-night cycle, or tidal cycles caused by the moon revolving around the earth.

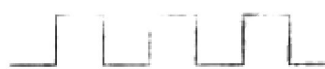
A periodic function is a function whose value repeats after a specific time interval. A periodic function is represented as :

$$f(x+p) = f(x)$$

Where "p" is the period of the function. For example, Sine wave, triangular wave, square wave, and saw tooth wave are periodic in nature.



Sine wave



Square wave



Rectangular wave



Saw tooth wave

All trigonometric functions repeat itself at regular intervals, or periods. The values of trigonometric functions for ' θ ' and ' $2n\pi \pm \theta$ ', where $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, are same. This periodic behavior of trigonometric functions is called periodicity.

$f(x)$	→	Periodic	, $p > 0$, $f(p+x) = f(x)$
"p"	→	Period of the function	

9.2.1 Periodicity of Sine Function

Suppose “ p ” is the period of Sine function, then:

$$\sin(\theta + p) = \sin \theta \dots\dots\dots(i) \quad \text{where } \theta \in R.$$

Now putting $\theta = 0$, we have:

$$\sin(0 + p) = \sin 0$$

$$\Rightarrow \sin p = 0$$

$$\Rightarrow p = \sin^{-1}(0) = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \dots$$



Key Facts
Period of a trigonometric function is the smallest +ve integer which when added to the original circular measure of the angle, gives the same value of the function.

Case 1: Put $p = \pi$ in (i), $\sin(\theta + \pi) = \sin \theta$, which is not true.

$$\therefore \sin(\pi + \theta) = -\sin \theta$$

$\therefore \pi$ is not the period of $\sin \theta$.

Case 2: Put $p = 2\pi$ in (i)

$$\sin(\theta + 2\pi) = \sin \theta \dots\dots\dots(ii)$$

$$\text{From (ii)} \quad \text{L.H.S} = \sin(2\pi + \theta) = \sin\left(2\pi + \theta\right) = \sin \theta = \text{R.H.S}$$

Since terminal side of angle is in first quadrant, and ‘ 2π ’ is the smallest +ve real number

for which: $\sin(\theta + 2\pi) = \sin \theta$.

$\therefore 2\pi$ is the period of $\sin \theta$.



Key Facts
Cosine is the periodic function and its period is 2π .

$$\cos(\theta + 2\pi) = \cos \theta$$

9.2.2 Periodicity of Tangent Function

Suppose “ p ” is the period of Tangent function, then:

$$\tan(\theta + p) = \tan \theta \dots\dots\dots(i) \quad \text{where } \theta \in R.$$

Now putting $\theta = 0$ in (i), we have:

$$\tan(0 + p) = \tan 0$$

$$\Rightarrow \tan p = 0$$

$$\Rightarrow p = \tan^{-1}(0) = 0, \pi, 2\pi, 3\pi, \dots$$

Put $p = \pi$ in (i)

$$\tan(\theta + \pi) = \tan \theta \dots\dots\dots(ii)$$

$$\text{From (ii): L.H.S} = \tan(\pi + \theta) = \tan\left(\pi + \theta\right) = \tan \theta = \text{R.H.S}$$

Since terminal side of an angle is in third quadrant, and ‘ π ’ is the least +ve real number for which:

$$\tan(\theta + \pi) = \tan \theta$$

$\therefore \pi$ is the period of $\tan \theta$.



Key Facts

Cotangent is the periodic function and its period is π .

$$\cot(\theta + \pi) = \cot \theta$$

If “ p ” is the period of a periodic function $f(x)$, then $\frac{1}{f(x)}$ is also a periodic function and will have the same period “ p ” as $f(x)$.

Thus, $y = \operatorname{cosec} \theta$ is a periodic function and its period is 2π because $\sin \theta = \frac{1}{\operatorname{cosec} \theta}$.

Similarly, $y = \sec \theta$ is a periodic function and its period is 2π because $\cos \theta = \frac{1}{\sec \theta}$.

If “ P ” is the period of the periodic function $f(x)$, then $f(ax + b)$, $a > 0$ is also a periodic function with a period $\frac{P}{|a|}$.

Trigonometric Functions	Period	Trigonometric Function	Period
$f(x) = \sin x$	2π	$f(x) = \sin ax$ or $f(x) = \sin(ax + b)$	$\frac{2\pi}{ a }$
$f(x) = \cos x$	2π	$f(x) = \cos ax$ or $f(x) = \cos(ax + b)$	$\frac{2\pi}{ a }$
$f(x) = \tan x$	π	$f(x) = \tan ax$ or $f(x) = \tan(ax + b)$	$\frac{\pi}{ a }$
$f(x) = \cot x$	π	$f(x) = \cot ax$ or $f(x) = \cot(ax + b)$	$\frac{\pi}{ a }$
$f(x) = \sec x$	2π	$f(x) = \sec ax$ or $f(x) = \sec(ax + b)$	$\frac{2\pi}{ a }$
$f(x) = \operatorname{cosec} x$	2π	$f(x) = \operatorname{cosec} ax$ or $f(x) = \operatorname{cosec}(ax + b)$	$\frac{2\pi}{ a }$

Example:

Find the periods of:

- (i) $f(x) = \sin 3x$ (ii) $f(x) = \cos \frac{2x}{5}$ (iii) $f(x) = \tan \frac{5x}{7}$

Solution:

- (i) We know that period of sine function is 2π , i. e., period of $\sin x = 2\pi$

Period of $\sin(ax + b) = \frac{2\pi}{|a|}$, where $a = 3$

$\Rightarrow$ Period of $\sin 3x = \frac{2\pi}{3}$

(ii) We know that period of $\cos x = 2\pi$

Period of $\cos(ax + b) = \frac{2\pi}{|a|}$, where $a = \frac{2}{5}$

$\Rightarrow$ Period of $\cos \frac{2x}{5} = \frac{2\pi}{\frac{2}{5}} = 2\pi \cdot \frac{5}{2} = 5\pi$.

Hence, 5π is period of $\cos \frac{2x}{5}$.

Check: Since ' 5π ' is period of $\cos \frac{2x}{5}$.

$$\therefore \cos\left(\frac{2x}{5} + 5\pi\right) = \cos\left(\frac{2}{5}(x + 2\pi)\right)$$

This clearly shows that ' 5π ' is period of $\cos \frac{2x}{5}$.

(iii) We know that the period of Tangent is π , i.e., Period of $\tan x = \pi$.

Period of $\tan(ax + b) = \frac{\pi}{|a|}$, where $a = \frac{5}{7}$

$\Rightarrow$ Period of $\tan \frac{5x}{7} = \frac{\pi}{\frac{5}{7}} = \pi \cdot \frac{7}{5} = \frac{7\pi}{5}$.

Hence, $\frac{7\pi}{5}$ is period of $\tan \frac{5x}{7}$.

Note: If " p " is the period of the periodic function $f(x)$ then $a f(x) + b$, $a > 0$, is also a periodic function with a period of " p ".

Trigonometric Functions	Period	Trigonometric Function	Period
$f(x) = a \sin x + b$	2π	$f(x) = a \operatorname{cosec} x + b$	2π
$f(x) = a \cos x + b$	2π	$f(x) = a \sec x + b$	2π
$f(x) = a \tan x + b$	π	$f(x) = a \cot x + b$	π

Example: Find the period of $f(x) = \cot 3x + \sin \frac{2x}{3}$.

Solution:

Period of $f(x) = \cot 3x$	Period of $f(x) = \sin \frac{2x}{3}$
Since, period of $\cot x = \pi$	Since, period of $\sin x = 2\pi$
Period of $\cot(ax + b) = \frac{\pi}{ a }$ where $a = 3$	Period of $\sin(ax + b) = \frac{2\pi}{ a }$ where $a = \frac{2}{3}$
Thus, the period of $\cot 3x = \frac{\pi}{3}$	Thus, the period of $\sin \frac{2x}{3} = \frac{2\pi}{\frac{2}{3}} = 3\pi$

Key Facts

- Periodicity of $\cos^n(x) = \frac{2\pi}{2} = \pi$
 $= \frac{\text{Periodicity of } \cos x}{2}$ (if ' n ' is even)
- Periodicity of $\cos^n(x)$
 $=$ Periodicity of $\cos x$
 $= 2\pi$ (if ' n ' is odd)
- Similarly, for Sine function.
- Periodicity of $\tan^n(x)$
 $=$ Periodicity of $\tan x$
 (no matter ' n ' is even or odd).

Hence,

$$\text{Period of } f(x) = \frac{\text{L.C.M of } \pi \text{ and } 3\pi}{\text{H.C.F of } 3 \text{ and } 1} = \frac{3\pi}{1} = 3\pi$$

Hence, 3π is a period of $\cot 3x + \sin \frac{2x}{3}$.

Example: Find the period of $f(x) = 7 \sin (3x + 5)$.

Solution: $f(x) = 7 \sin (3x + 5)$

Since, period of $\sin x = 2\pi$

$$\text{Period of } \sin (ax + b) = \frac{2\pi}{|a|} \text{ where } a = 3$$

Thus, the period of $7 \sin (3x + 5) = \frac{2\pi}{3}$.

9.3 Maximum and Minimum Values of Trigonometric Functions

(i) $a + b \sin \theta$

(ii) $a + b \cos \theta$

(iii) $a + b \sin (c\theta + d)$

(iv) $a + b \cos (c\theta + d)$

The reciprocal of the above where a, b, c and d are real numbers.

(i) $a + b \sin \theta$

$$\text{Maximum value (M)} = a + |b| (1)$$

$$= a + |b| \Rightarrow \text{when } \sin \theta = 1$$

or $a + b \sin \theta$ is maximum when $b \sin \theta$ is maximum and it is maximum when $\sin \theta$ is maximum.

As $\sin \theta$ is maximum when $\sin \theta = 1$.

$$\text{Minimum value (m)} = a + |b| (-1)$$

$$= a - |b| \Rightarrow \text{when } \sin \theta = -1$$

or $a + b \sin \theta$ is minimum when $b \sin \theta$ is minimum and it is minimum when $\sin \theta$ is minimum.

We know that $\sin \theta$ is minimum when $\sin \theta = -1$.

(ii) $a + b \cos \theta$

$$\text{Maximum value (M)} = a + |b| (1)$$

$$= a + |b| \Rightarrow \text{when } \cos \theta = 1$$

or $a + b \cos \theta$ is maximum when $b \cos \theta$ is maximum and it is maximum when $\cos \theta$ is maximum.

As $\cos \theta$ is maximum when $\cos \theta = 1$.

$$\text{Minimum value (m)} = a + |b| (-1)$$

$$= a - |b| \Rightarrow \text{when } \cos \theta = -1$$

or $a + b \cos \theta$ is minimum when $b \cos \theta$ is minimum and it is minimum when $\cos \theta$ is minimum.

As $\cos \theta$ is minimum when $\cos \theta = -1$.

(iii) $a + b \sin (c\theta + d)$

$$\text{Maximum value (M)} = a + |b| (1)$$

$$= a + |b| \Rightarrow \text{when } \sin (c\theta + d) = 1$$

or $a + b \sin (c\theta + d)$ is maximum when $b \sin (c\theta + d)$ is maximum and it is maximum when $\sin (c\theta + d)$ is maximum. As $\sin (c\theta + d)$ is maximum when $\sin (c\theta + d) = 1$.

$$\begin{aligned}\text{Minimum value (m)} &= a + |b| (-1) \\ &= a - |b| \Rightarrow \text{when } \sin (c\theta + d) = -1\end{aligned}$$

or $a + b \sin (c\theta + d)$ is minimum when $b \sin (c\theta + d)$ is minimum and it is minimum when $\sin (c\theta + d)$ is minimum. As $\sin (c\theta + d)$ is minimum when $\sin (c\theta + d) = -1$.

(iv) $a + b \cos (c\theta + d)$

$$\begin{aligned}\text{Maximum value (M)} &= a + |b| (1) \\ &= a + |b| \Rightarrow \text{when } \cos (c\theta + d) = 1\end{aligned}$$

or $a + b \cos (c\theta + d)$ is maximum when $b \cos (c\theta + d)$ is maximum and it is maximum when $\cos (c\theta + d)$ is maximum. As $\cos (c\theta + d)$ is maximum when $\cos (c\theta + d) = 1$.

$$\begin{aligned}\text{Minimum value (m)} &= a + |b| (-1) \\ &= a - |b| \Rightarrow \text{when } \cos (c\theta + d) = -1\end{aligned}$$

or $a + b \cos (c\theta + d)$ is minimum when $b \cos (c\theta + d)$ is minimum and it is minimum when $\cos (c\theta + d)$ is minimum. As $\cos (c\theta + d)$ is minimum when $\cos (c\theta + d) = -1$.

Example: Find the maximum and minimum values of $y = 3 + 4 \sin \theta$.

Solution: $y = 3 + 4 \sin \theta$ (i)

Since, $y = a + b \sin \theta$ (ii)

Comparing coefficients of (i) and (ii):

$$a = 3 \quad \text{and} \quad b = 4$$

$$\text{Maximum value (M)}$$

$$a + |b| = 3 + 4 = 7$$

$$M = 7$$

$$\text{Minimum value (m)}$$

$$a - |b| = 3 - 4 = -1$$

$$m = -1$$

Example: Find the maximum and minimum values of $y = \frac{1}{3} - 5 \cos \theta$.

Solution: $y = \frac{1}{3} - 5 \cos \theta$ (i)

Since, $y = a + b \cos \theta$ (ii)

Comparing coefficients of (i) and (ii):

$$a = \frac{1}{3} \quad \text{and} \quad b = -5$$

$$\text{Maximum value (M)}$$

$$a + |b| = \frac{1}{3} + |-5|$$

$$= \frac{1}{3} + 5 = \frac{16}{3}$$

$$M = \frac{16}{3}$$

$$\text{Minimum value (m)}$$

$$a - |b| = \frac{1}{3} - |-5|$$

$$= \frac{1}{3} - 5 = \frac{-14}{3}$$

$$m = \frac{-14}{3}$$

Example: Find the maximum and minimum values of the following trigonometric functions.

(i) $y = 1 + 2 \sin \theta$ (ii) $y = 3 + 2 \cos (3\theta - 2)$ (iii) $y = \frac{1}{1 + 3 \sin (2\theta - 15)}$

Solution: (i) $y = 1 + 2 \sin \theta$(i)

Since, $y = a + b \sin \theta$(ii)

Comparing coefficients of (i) and (ii):

$$a = 1 \text{ and } b = 2$$

Maximum value (M)

$$a + |b| = 1 + 2 = 3$$

$$M = 3$$

Minimum value (m)

$$a - |b| = 1 - 2 = -1$$

$$m = -1$$

(ii) $y = 3 + 2 \cos (3\theta - 2)$(i)

Since, $y = a + b \cos (c\theta + d)$(ii)

Comparing coefficients of (i) and (ii)

$$a = 3 \text{ and } b = 2$$

Maximum value (M)

$$a + |b| = 3 + 2 = 5$$

$$M = 5$$

Minimum value (m)

$$a - |b| = 3 - 2 = 1$$

$$m = 1$$

(iii) $y = \frac{1}{1 + 3 \sin (2\theta - 15)}$(i)

Then the corresponding reciprocal function of (i) is

$$y = 1 + 3 \sin (2\theta - 15)$$
.....(ii)

Since, $y = a + b \sin (c\theta + d)$ (iii)

Comparing coefficients of (ii) and (iii):

$$a = 1 \text{ and } b = 3$$

Maximum value (M)

$$a + |b| = 1 + 3 = 4$$

$$M = 4 > 0$$

Minimum value (m)

$$a - |b| = 1 - 3 = -2$$

$$m = -2 < 0$$

Now for maximum value of (i)

Let

$$M' = \frac{1}{M} = \frac{1}{4}$$

$$M' = \frac{1}{4}$$

Note: If $M > 0$ and $m > 0$, then

$$M' = \frac{1}{m} \text{ and}$$

Now for minimum value of (i)

Let

$$m' = \frac{1}{m} = \frac{1}{-2}$$

$$m' = \frac{1}{-2}$$

$$m' = \frac{1}{M'}$$

9.4 Graphs of Trigonometric Functions

All the trigonometric functions are periodic functions. We can draw their graphs on the intervals of lengths equal to their periods. Because, when the graph of a periodic function of period 'p' is drawn in a given interval, then it is sufficient to draw its graph only in that interval. Further, it can easily be drawn completely by repeating it over the intervals of lengths 'p'.

Procedure for Sketching Graphs of Sine and Cosine Functions

To graph $y = a \sin bx$ or $y = a \cos bx$, with $b > 0$, follow these steps.

Step 1: Find the period $\frac{2\pi}{b}$. Initially, start from "0" on the x-axis, and lay off a distance of $\frac{2\pi}{b}$.

Step 2: Divide the given interval into four equal parts:

- Find the midpoint of the given interval by adding the end-points of each interval and dividing the sum by 2.
- Find the quarter points (the midpoints of the two intervals obtained in part (a)).
- Continue in the same way until the required number of equal parts are obtained.

Step 3: Evaluate the function for each of the five/nine/thirteen x-values resulting from step 2.

The resulting points will be maximum points, minimum points, and x-intercepts.

Step 4: Plot the points obtained in step 3 and join them with a sinusoidal curve having amplitude $|a|$.

Step 5: Draw the graph over additional periods if required.

Angles in Degrees and Radians												
0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°

Graph of $y = \sin x$

We know that $\sin x$ is a periodic function of period 2π .

Step 1: For this function $b = 1$, so the period is 2π .

The function will be graphed over the interval $[0, 2\pi]$.

Step 2: Divides the interval $[0, 2\pi]$ in twelve equal parts to obtain the x-values:

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$$

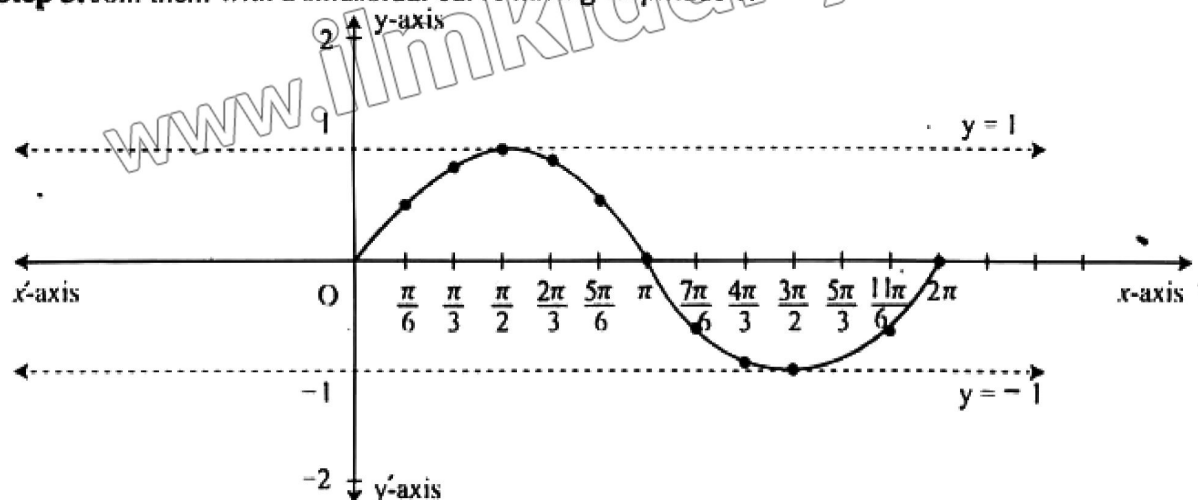
Step 3: For this, we first construct the table determined by the x-values in Step 2.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
Sin x	0	0.5	0.86	1	0.86	0.5	0	-0.5	-0.86	-1	-0.86	-0.5	0

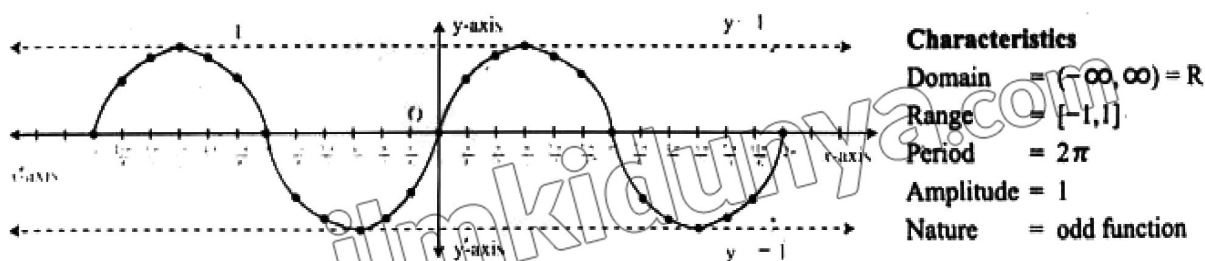
Step 4: Plot the points $(0, 0)$, $(\frac{\pi}{6}, 0.5)$, $(\frac{\pi}{3}, 0.86)$, $(\frac{\pi}{2}, 1)$, $(\frac{2\pi}{3}, 0.86)$, $(\frac{5\pi}{6}, 0.5)$,

$$(\pi, 0), (\frac{7\pi}{6}, -0.5), (\frac{4\pi}{3}, -0.86), (\frac{3\pi}{2}, -1), (\frac{5\pi}{3}, -0.86), (\frac{11\pi}{6}, -0.5), (2\pi, 0).$$

Step 5: Join them with a sinusoidal curve having amplitude 1.



Step 6: Extend the graph by repeating the cycle, from 0 to -2π .



The graph in the interval $[0, 2\pi]$ is called a cycle. Since the period of the Sine function is 2π . So, the sine graph can be extended on both side of x -axis through every interval of 2π .

Graph of $y = \cos x$

We know that $\cos x$ is a periodic function of period 2π .

Step 1: For this function $b = 1$, so the period is 2π . The function will be graphed over the interval $[0, 2\pi]$.

Step 2: Divides the interval $[0, 2\pi]$ in twelve equal parts to obtain the x -values:

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$$

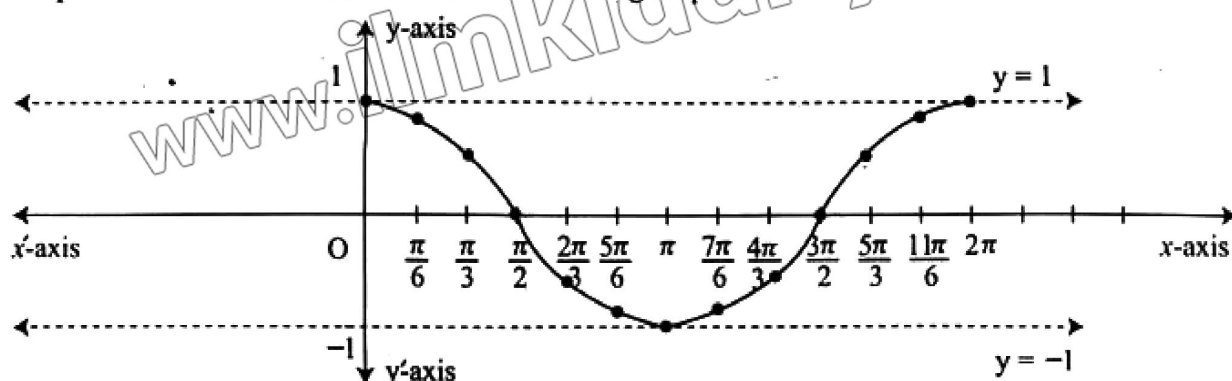
Step 3: For this, we first construct the table determined by the x -values in Step 2.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	0.86	0.5	0	-0.5	-0.86	-1	-0.86	-0.5	0	0.5	0.86	1

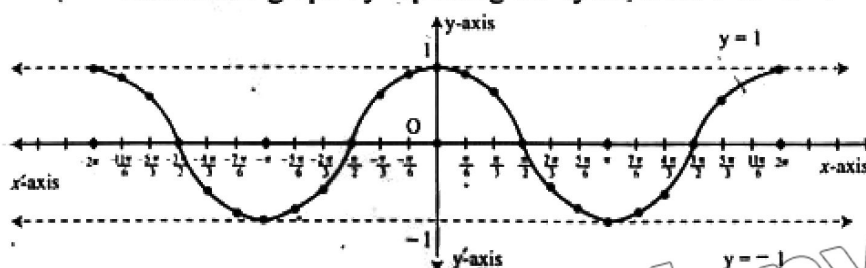
Step 4: Plot the points $(0, 0)$, $(\frac{\pi}{6}, 0.86)$, $(\frac{\pi}{3}, 0.5)$, $(\frac{\pi}{2}, 0)$, $(\frac{2\pi}{3}, -0.5)$, $(\frac{5\pi}{6}, -0.86)$,

$$(\pi, -1), (\frac{7\pi}{6}, -0.86), (\frac{4\pi}{3}, -0.5), (\frac{3\pi}{2}, 0), (\frac{5\pi}{3}, 0.5), (\frac{11\pi}{6}, 0.86), (2\pi, 1).$$

Step 5: Join them with a sinusoidal curve having amplitude 1.



Step 6: Extend the graph by repeating the cycle, from 0 to -2π .



Characteristics

Domain = $(-\infty, \infty) = \mathbb{R}$

Range = $[-1, 1]$

Period = 2π

Amplitude = 1

Nature = even function

The graph in the interval $[0, 2\pi]$ is called a cycle. Since the period of the Cosine function is 2π . So, the Cosine graph can be extended on both side of x -axis through every interval of 2π .

Graph of $y = \tan x$

We know that $\tan x$ is a periodic function of period π .

Step 1: For this function $b = 1$, so the period is π . The function will be graphed over the interval $[0, \pi]$.

Step 2: Divides the interval $[0, \pi]$ in twelve equal parts (From 0 to $\pm\pi$) to obtain the x -values:

$$0, \pm\frac{\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pm\frac{5\pi}{6}, \pm\pi$$

Initially, we draw the graph of tangent function from 0 to $\pm\pi$.

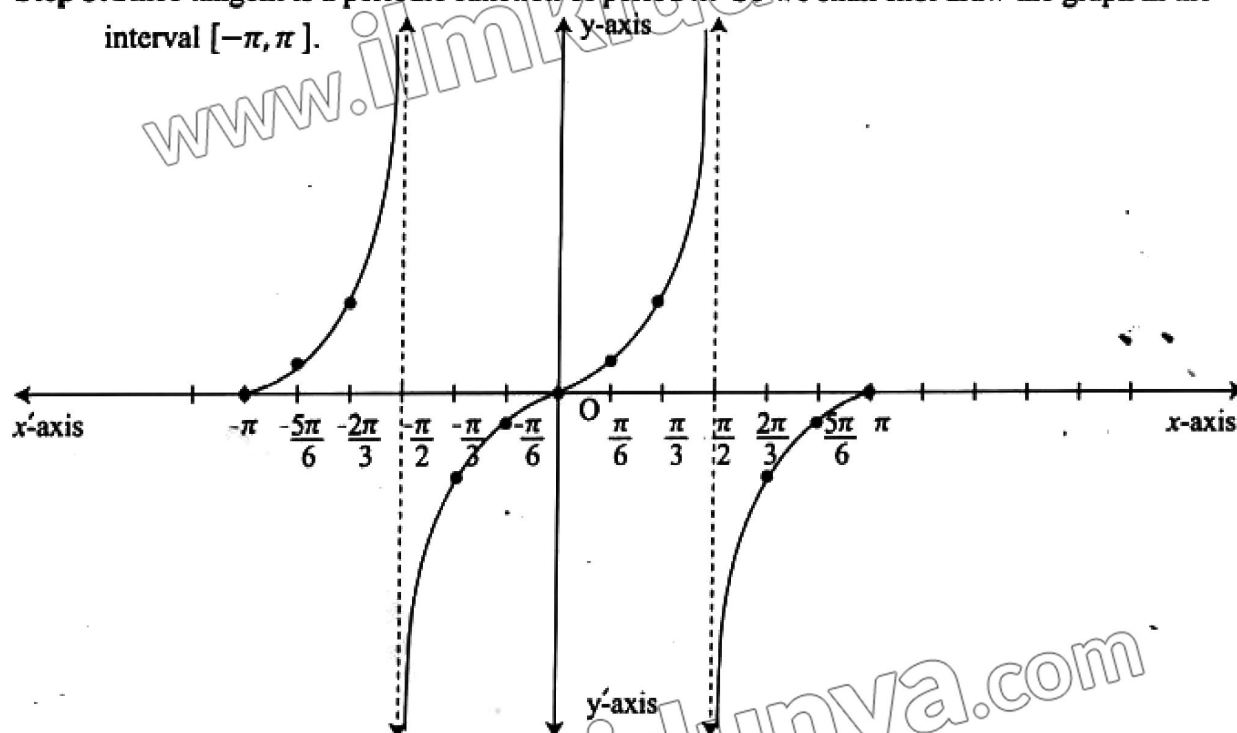
Step 3: For this, we first construct the table determined by the x -values in Step 2.

x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\tan x$	0	0.58	1.73	$-\infty$	-1.73	-0.58	0	0.58	1.73	∞	-1.73	-0.58	0

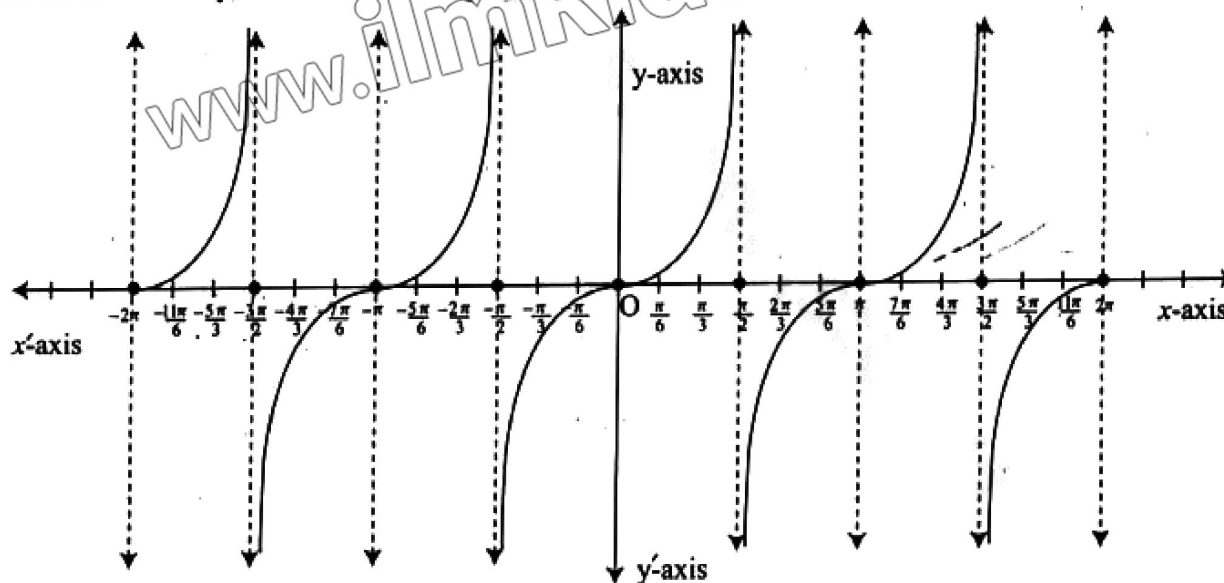
Step 4: Plot the points $(-\pi, 0), (-\frac{5\pi}{6}, 0.58), (-\frac{2\pi}{3}, 1.73), (-\frac{\pi}{2}, -\infty), (-\frac{\pi}{3}, -1.73), (-\frac{\pi}{6}, -0.58), (0, 0), (\frac{\pi}{6}, 0.58), (\frac{\pi}{3}, 1.73), (\frac{\pi}{2}, \infty), (\frac{2\pi}{3}, -1.73), (\frac{5\pi}{6}, -0.58), (\pi, 0)$.

Join these points by a free hand curve to obtain the graph of $\tan x$.

Step 5: Since tangent is a periodic function of period π . So we shall first draw the graph in the interval $[-\pi, \pi]$.



In the similar way, we can draw the graphs for the interval from -2π to 2π .



Characteristics

Domain	=	$\{x: x \in \mathbb{R} \text{ and } x \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\}$
Range	=	$(-\infty, \infty) = \mathbb{R}$
Period	=	π
Amplitude	=	nil
Nature	=	odd function

The graph in the interval $[0, \pi]$ is called a cycle. Since the period of the Tangent function is π . So, the Tangent can be extended on both side of x -axis through every interval of π .

Graph of $y = 3 \sin 2x$

We know that $\sin x$ is a periodic function of period 2π .

Step 1: For this function $b = 2$, so the period of $y = 3 \sin 2x$ is $\frac{2\pi}{2} = \pi$. The function will be graphed over the interval $[0, \pi]$.

Step 2: Divides the interval $[0, \pi]$ in eight equal parts to obtain the x-values:

$$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi$$

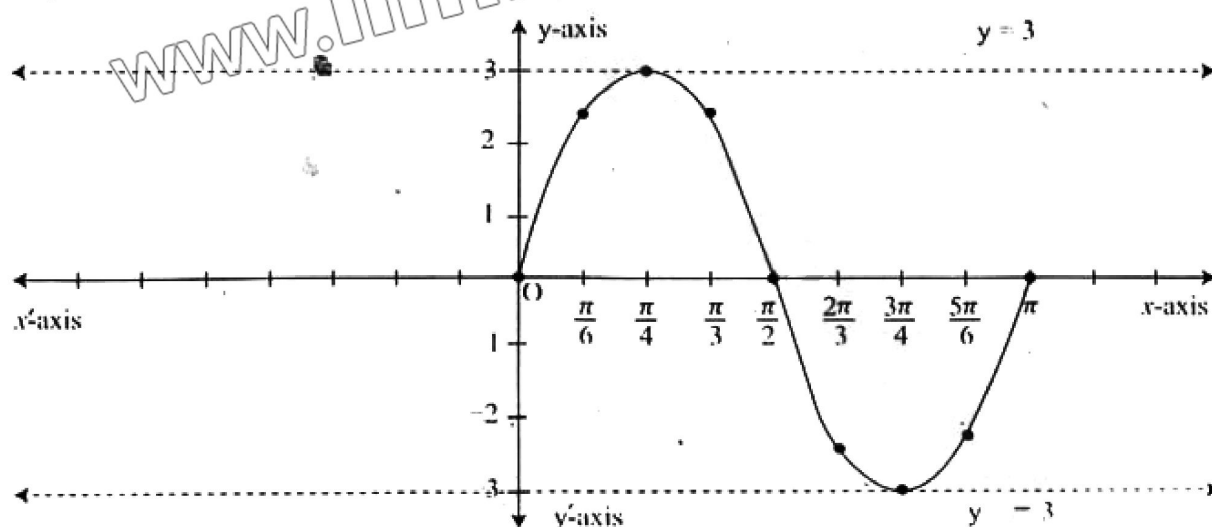
Step 3: For this, we first construct the table determined by the x-values in Step 2.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$2x$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2x$	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	0
$3 \sin 2x$	0	2.61	3	2.61	0	-2.61	-3	-2.61	0

Step 4: Plot the points $(0, 0)$, $(\frac{\pi}{6}, 2.61)$, $(\frac{\pi}{4}, 3)$, $(\frac{\pi}{3}, 2.61)$, $(\frac{\pi}{2}, 0)$, $(\frac{2\pi}{3}, -2.61)$, $(\frac{3\pi}{4}, -3)$, $(\frac{5\pi}{6}, -2.61)$, $(\pi, 0)$.

Join them with a sinusoidal curve having amplitude 3.

Step 5: Extend the graph by repeating the cycle, from 0 to π .



In the similar way, we can draw the graphs for the interval from -2π to 2π .

Characteristics

Domain = $(-\infty, \infty) = \mathbb{R}$
 Range = $[-3, 3]$
 Period = π
 Amplitude = 3
 Nature = odd function

Key Facts

For $y = a \sin bx$, we know that $-1 \leq \sin bx \leq 1$ and $-a \leq a \sin bx \leq a$. Also, $\sin x$ is a periodic function of period 2π . Therefore, $\sin bx$ is periodic function of period $\frac{2\pi}{b}$. Hence, $a \sin bx$ is periodic function of period $\frac{2\pi}{b}$.



Key Facts

For $y = a \cos bx$, we know that $-1 \leq \cos bx \leq 1$ and $-a \leq a \cos bx \leq a$. Also, $\cos x$ is a periodic function of period 2π . Therefore, $\cos bx$ is periodic function of period $\frac{2\pi}{b}$. Hence, $a \cos bx$ is periodic function of period $\frac{2\pi}{b}$.



Graph of $y = 3 \cos 2x$

We know that $\cos x$ is a periodic function of period 2π .

Step 1: For this function $b = 2$, so the period of $y = 3 \cos 2x$ is $\frac{2\pi}{2} = \pi$. The function will be graphed over the interval $[0, \pi]$.

Step 2: Divides the interval $[0, \pi]$ in eight equal parts to obtain the x-values:

$$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi$$

Step 3: For this, we first construct the table determined by the x-values in Step 2.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$2x$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos 2x$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$3 \cos 2x$	3	1.5	0	-1.5	-3	-1.5	0	1.5	3

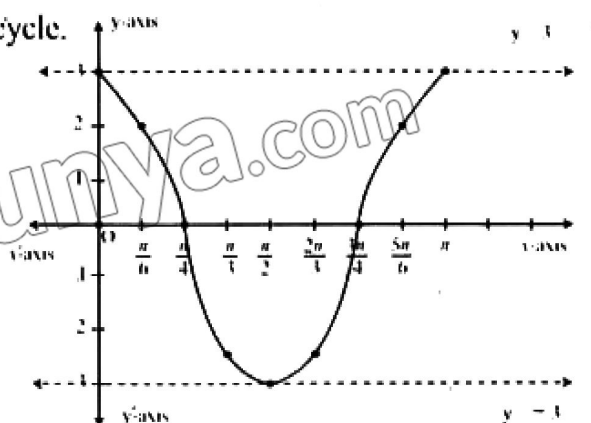
Step 4: Plot the points $(0, 3)$, $(\frac{\pi}{6}, 1.5)$, $(\frac{\pi}{4}, 0)$, $(\frac{\pi}{3}, -1.5)$, $(\frac{\pi}{2}, -3)$, $(\frac{2\pi}{3}, -1.5)$, $(\frac{3\pi}{4}, 0)$, $(\frac{5\pi}{6}, 1.5)$, $(\pi, 3)$.

Join them with a sinusoidal curve having amplitude 3.

Step 5: The graph can be extended by repeating the cycle.

Characteristics

Domain = $(-\infty, \infty) = \mathbb{R}$
 Range = $[-3, 3]$
 Period = π
 Amplitude = 3
 Nature = even function



Graph of $y = \operatorname{cosec} x$

Step 1: For this function $b = 1$, so the period is 2π . The function will be graphed over the interval $[0, 2\pi]$.

$$0, \pm \frac{\pi}{6}, \pm \frac{\pi}{3}, \pm \frac{\pi}{2}, \pm \frac{2\pi}{3}, \pm \frac{5\pi}{6}, \pm \pi$$

Step 3: For this, we first construct the table determined by the x-values in Step 2.

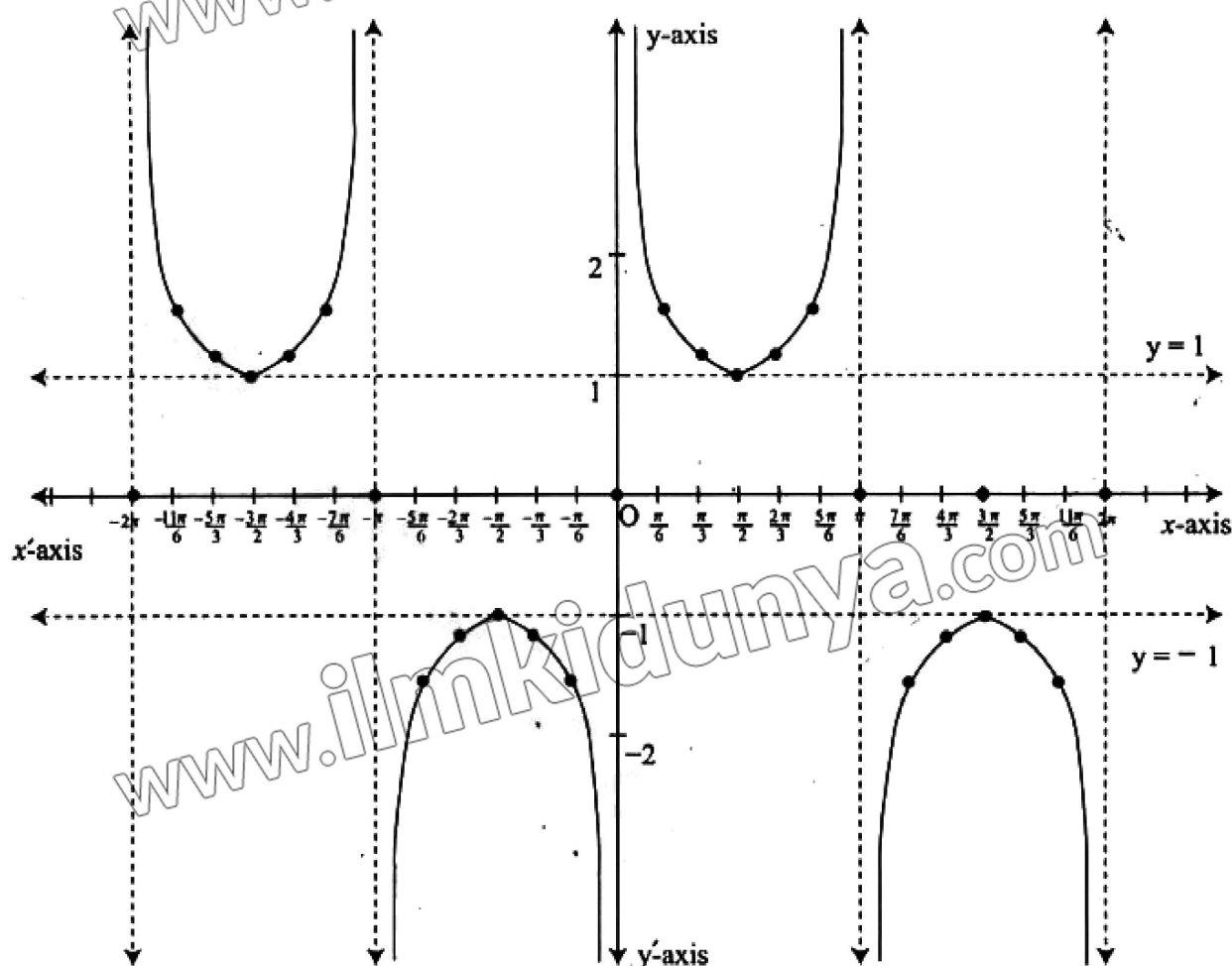
x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\operatorname{Cosec} x$	$-\infty$	-2	-1.15	-1	-1.15	-2	∞	2	1.15	1	1.15	2	∞

$$(-\pi, -\infty), \left(-\frac{5\pi}{6}, -2\right), \left(-\frac{2\pi}{3}, -1.15\right), \left(-\frac{\pi}{2}, -1\right), \left(-\frac{\pi}{3}, -1.15\right), \left(-\frac{\pi}{6}, -2\right), (0, \infty),$$

$$\left(\frac{\pi}{6}, 2\right), \left(\frac{\pi}{3}, 1.15\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{2\pi}{3}, 1.15\right), \left(\frac{5\pi}{6}, 2\right), \text{ and } (\pi, \infty).$$

Step 5: Extend the graph by repeating the cycle.

In the similar way, we can draw the graphs for the interval from -2π to 2π .



The graph in the interval $[0, 2\pi]$ is called a cycle. Since the period of the cosecant function is 2π . So, the cosecant graph can be extended on both side of x -axis through every interval of 2π .

Characteristics

Domain	=	$\{x : x \in \mathbb{R} \text{ and } x \neq n\pi, n \text{ an integer}\}$
Range	=	$(-\infty, -1] \cup [1, \infty)$ or $ y \geq 1$
Period	=	2π
Amplitude	=	nil
Nature	=	odd function

Graph of $y = \sec x$

We know that Secant function is a reciprocal of the Cosine function which is a periodic function of period 2π . Therefore, Secant is also a periodic function of period 2π .

Step 1: For this function $b = 1$, So the period is 2π . The function will be graphed over the interval $[0, 2\pi]$.

Step 2: Divides the interval $[0, 2\pi]$ in twelve equal parts (From 0 to $\pm\pi$) to obtain the x-values:

$$0, \pm\frac{\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pm\frac{5\pi}{6}, \pm\pi$$

Initially, we draw the graph of Cosecant function from 0 to $\pm\pi$.

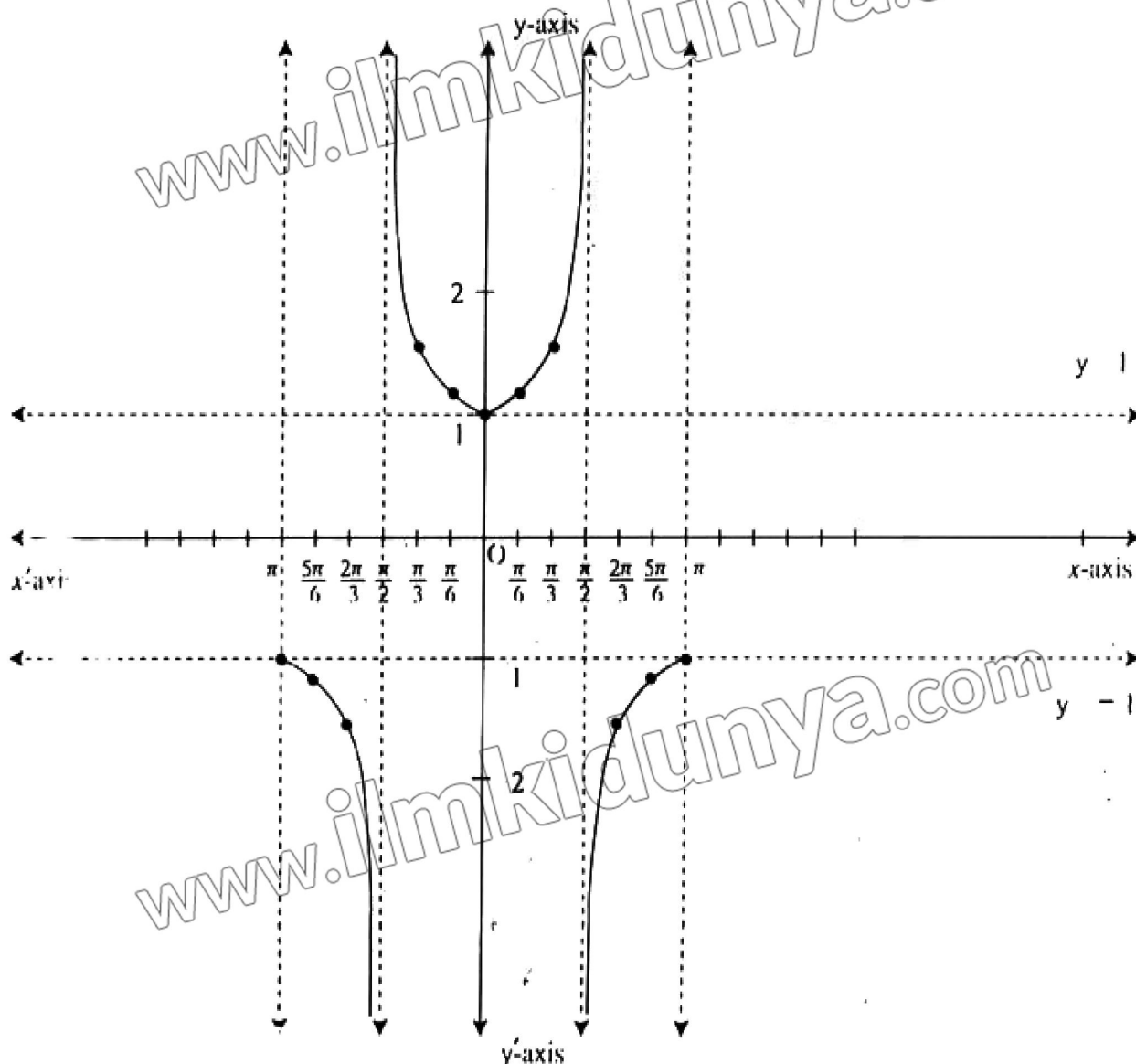
Step 3: For this, we first construct the table determined by the x-values in Step 2.

x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\sec x$	-1	-1.15	-2	∞	2	1.15	1	1.15	2	∞	-2	-1.15	-1

Step 4: Plot the points $(-\pi, -1), (-\frac{5\pi}{6}, -1.15), (-\frac{2\pi}{3}, -2), (-\frac{\pi}{2}, \infty), (-\frac{\pi}{3}, 2), (-\frac{\pi}{6}, 1.15)$

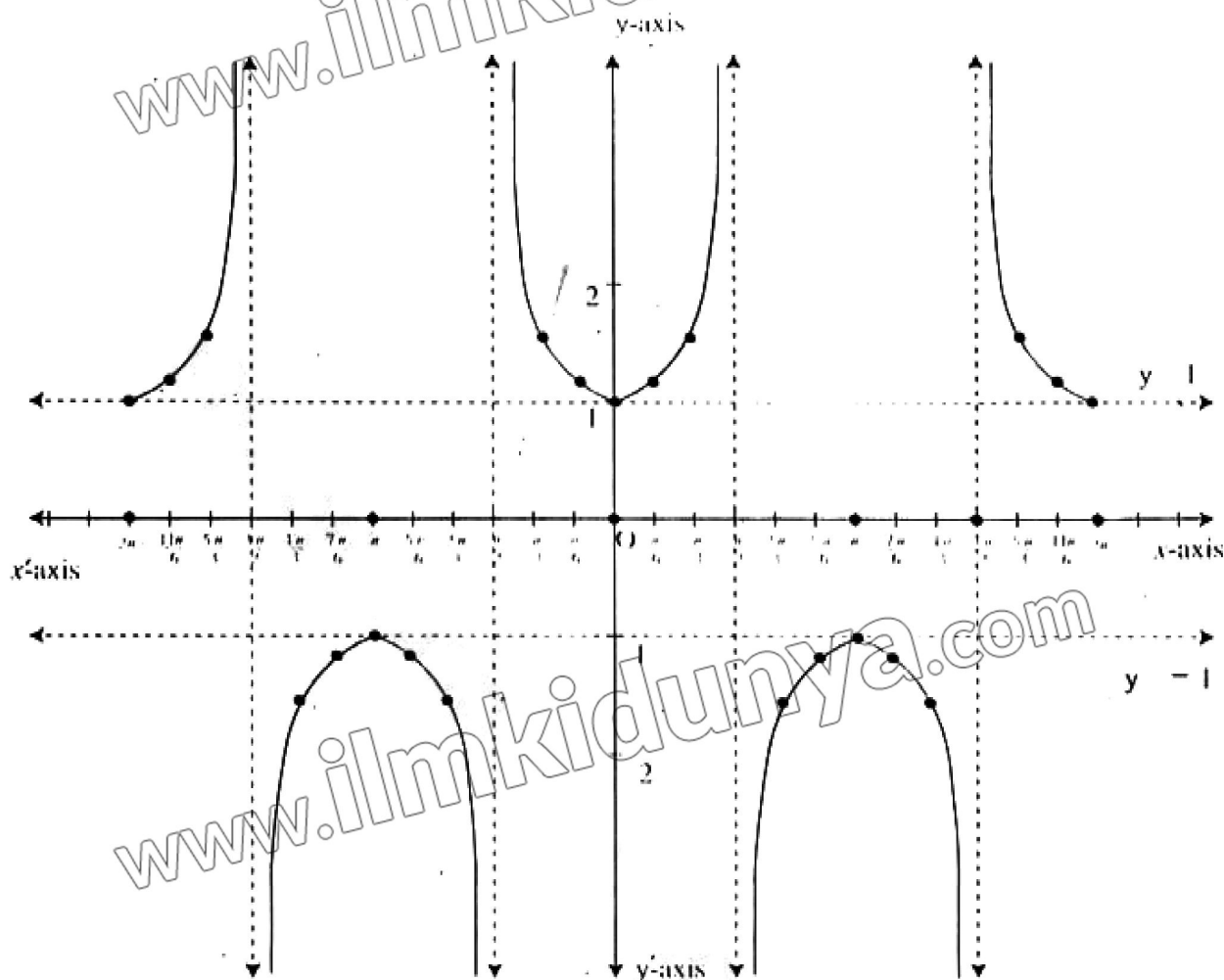
$(0, 1), (\frac{\pi}{6}, 1.15), (\frac{\pi}{3}, 2), (\frac{\pi}{2}, \infty), (\frac{2\pi}{3}, -2), (\frac{5\pi}{6}, -1.15), (\pi, -1).$

Join these points by a free hand curve to obtain the graph of $\sec x$.



Step 5: The graph can be extended by repeating the cycle.

In the similar way, we can draw the graphs for the interval from -2π to 2π .



The graph in the interval $[0, 2\pi]$ is called a cycle. Since the period of the cosecant function is 2π . So, the cosecant graph can be extended on both side of x-axis through every interval of 2π .

Characteristics

Domain	=	$\{x: x \in \mathbb{R} \text{ and } x \neq \frac{\pi}{2} + n\pi, n \text{ an integer}\}$
Range	=	$(-\infty, -1] \cup [1, \infty)$ or $ y \geq 1$
Period	=	2π
Amplitude	=	Nil
Nature	=	even function

Graph of $y = \cot x$

We know that $\cot x$ is a periodic function of period π .

Step 1: For this function $b = 1$, so the period is π . The function will be graphed over the interval $[0, \pi]$.

Step 2: Divides the interval $[0, \pi]$ in twelve equal parts (from 0 to $\pm\pi$) to obtain the x-values:

$$0, \pm\frac{\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pm\frac{5\pi}{6}, \pm\pi$$

Step 3: For this, we first construct the table determined by the x-values in Step 2.

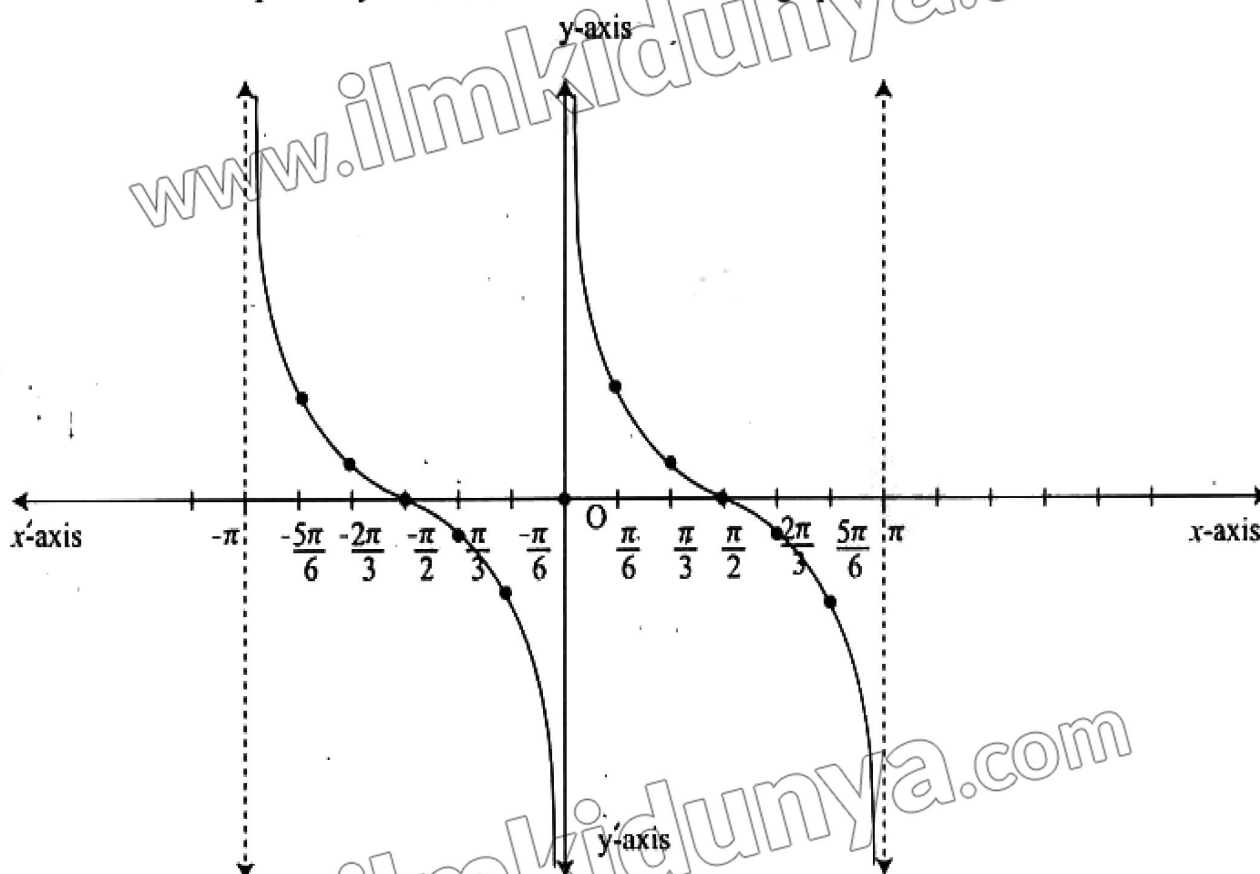
x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cot x$	$-\infty$	1.73	0.58	$-\infty$	-0.58	-1.73	0	1.73	0.58	∞	-0.58	-1.73	∞

Step 4: Plot the points

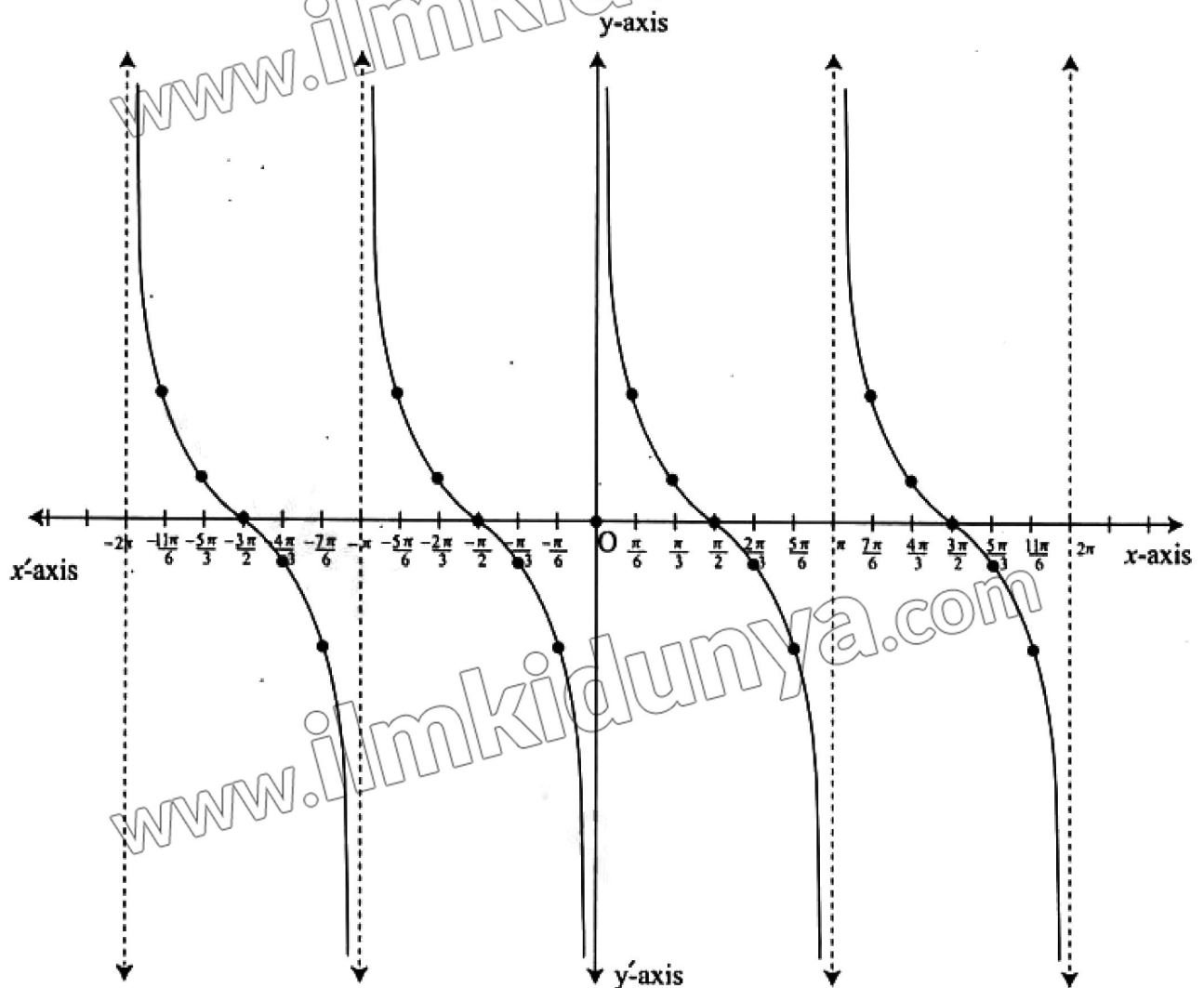
$$(-\pi, 0), \left(-\frac{5\pi}{6}, 0.58\right), \left(-\frac{2\pi}{3}, 1.73\right), \left(-\frac{\pi}{2}, \infty\right), \left(-\frac{\pi}{3}, -0.58\right), \left(-\frac{\pi}{6}, -1.73\right), (0, 0),$$

$$\left(\frac{\pi}{6}, 1.73\right), \left(\frac{\pi}{3}, 0.58\right), \left(\frac{\pi}{2}, \infty\right), \left(\frac{2\pi}{3}, -1.73\right), \left(\frac{5\pi}{6}, -0.58\right), (\pi, 0).$$

Join these points by a free hand curve to obtain the graph of $\cot x$.



Step 5: Since cotangent is a periodic function of period π . So we shall first draw the graph in the interval $[-\pi, \pi]$. In the similar way, we can draw the graphs for the interval from -2π to 2π .



The graph in the interval $[0, \pi]$ is called a cycle. Since the period of the cotangent function is π . So, the cotangent graph can be extended on both side of x -axis through every interval of π .

Characteristics

Domain	=	$\{x: x \in \mathbb{R} \text{ and } x \neq n\pi, n \text{ an integer}\}$
Range	=	$(-\infty, \infty) = \mathbb{R}$
Period	=	π
Amplitude	=	nill
Nature	=	odd function

9.5 Even and Odd Trigonometric Functions

All functions, including trigonometric functions, can be categorized as even, odd or neither.

- A function is odd if and only if $f(-x) = -f(x)$ and is symmetric (by reflection) with respect to the origin.
- A function is even if and only if $f(-x) = f(x)$ and is symmetric (by 180° rotation) with respect to the y-axis.

Key Facts



- The graph of Sine function is symmetric about the origin therefore, it is an odd function.
- The graph of Cosine function is symmetric with respect to y-axis therefore it is an even function.

Example :

Check whether the following function is odd or even?

Solution: $f(x) = x^3 \cdot \sin x$ (i)

Replace 'x' by '-x'

$$\begin{aligned} f(-x) &= (-x)^3 \cdot \sin(-x) \\ &= -(x)^3 \cdot (-\sin(x)) \\ &= x^3 \cdot \sin x = f(x) \end{aligned}$$

$$\text{As } f(-x) = f(x)$$

So (i) is an even function.

Example:

Check whether the following function is odd or even?

$$(i) \quad y = 3 \sin x + 4 \cos x \quad (ii) \quad y = \frac{\tan x}{x - \sin x}$$

Solution:

$$(i) \quad y = 3 \sin x + 4 \cos x$$

Replace 'x' by '-x'

$$f(-x) = 3 \sin(-x) + 4 \cos(-x)$$

$$= -3 \sin x + 4 \cos x$$

$$= -(3 \sin x + 4 \cos x)$$

$$\text{As, } f(-x) \neq f(x) \text{ or } f(-x) \neq -f(x)$$

So, $f(x)$ is neither even nor odd function.

(ii) $y = \frac{\tan x}{x - \sin x}$

Replace 'x' by '-x'

$$f(-x) = \frac{\tan(-x)}{(-x) - \sin(-x)}$$

$$= \frac{-\tan x}{-x + \sin x}$$

$$= \frac{\tan x}{x - \sin x}$$

$$\text{As, } f(-x) = f(x)$$

So, $f(x)$ is an even function.

Key Facts

O	$\sin(-x)$	=	$-\sin x$
E	$\cos(-x)$	=	$\cos x$
O	$\tan(-x)$	=	$-\tan x$
O	$\cot(-x)$	=	$-\cot x$
E	$\sec(-x)$	=	$\sec x$
O	$\operatorname{cosec}(-x)$	=	$-\operatorname{cosec} x$

9.6 Application

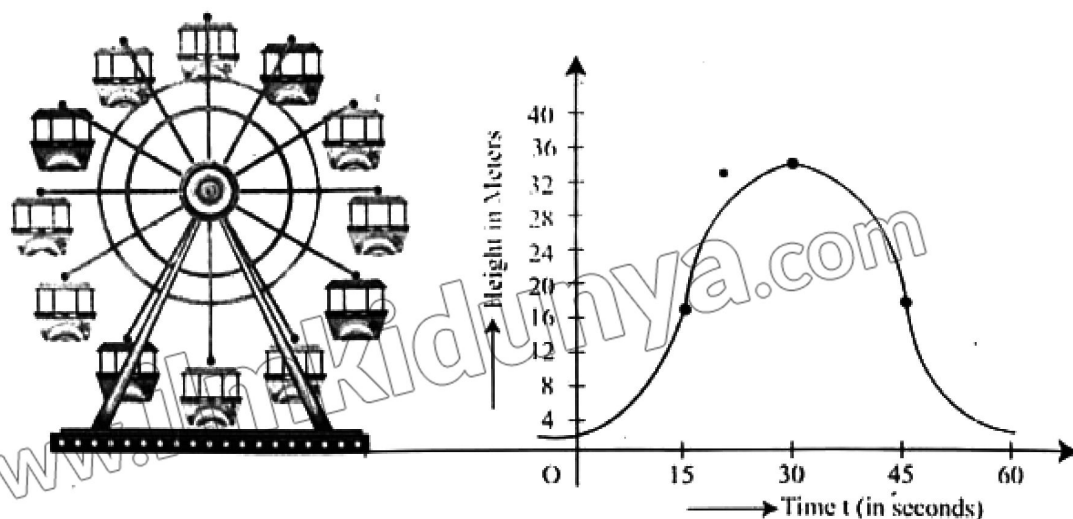
Example:

A Ferris wheel has a diameter of 30 m with a center 17 m above the ground. It makes one complete revolution every 60 seconds.

- (1) Graph one complete period of the graph that models the height in relation to time.
Assume a rider starts at the lowest point.
- (2) Find the equation of the graph using the cosine function.
- (3) What is the height of the rider at 45 seconds?
- (4) At what time or times, the rider is at a height of 15 m?

Solution:

(1)



(2) Since

$$y = A \cos(Bx - D) + C$$
$$h(t) = A \cos(Bt) + C \quad (i)$$

Where,

$$|A| = 15 \Rightarrow A = -15 \quad \text{and } \frac{2\pi}{B} \text{ is the period.}$$

$$60 = \frac{2\pi}{B} \Rightarrow B = \frac{\pi}{30} \text{ and } C = 17$$

Putting values in the above equation (i)

$$h(t) = -15 \cos\left(\frac{\pi}{30} t\right) + 17$$
$$= -15 \cos\left(\frac{\pi t}{30}\right) + 17$$

This is the equation which will give us the height of the rider at any time 't'.

(3) Since

$$h(t) = A \cos(Bt) + C$$
$$h(45) = -15 \cos\left(\frac{45\pi}{30}\right) + 17$$
$$= -15 \cos\left(\frac{3\pi}{2}\right) + 17$$
$$= 17 \text{ meters}$$

(4)

$$h(t) = -15 \cos\left(\frac{\pi}{30} t\right) + 17$$

$$15 = -15 \cos\left(\frac{\pi}{30} t\right) + 17$$

$$-2 = -15 \cos\left(\frac{\pi}{30} t\right)$$

$$\cos\left(\frac{\pi t}{30}\right) = \frac{2}{15}$$

$$\frac{\pi t}{30} = \cos^{-1}\left(\frac{2}{15}\right)$$

$$t = \frac{30}{\pi} (1.437)$$

Here,

$$\theta_1 = 1.437, \text{ and } \theta_2 = 2\pi - 1.43 = 4.846$$

$$t_1 = \frac{30}{\pi} (1.437), \text{ and } t_2 = \frac{30}{\pi} (4.846)$$

$$= 13.7 \text{ seconds}$$

$$= 46.3 \text{ seconds}$$

Exercise 9.1

1. Find the maximum and minimum values of the following trigonometric functions.

(i) $y = 2 - 2 \cos \theta$

(ii) $y = \frac{2}{3} - \frac{1}{2} \sin \theta$

(iii) $y = \frac{1}{5} - 2 \sin (3\theta - 7)$

(iv) $y = 7 + \frac{3}{5} \cos (2\theta - 1)$

2. Find the maximum and minimum values of the following reciprocal trigonometric functions.

(i) $y = \frac{1}{4 + 3 \sin \theta}$

(ii) $y = \frac{1}{\frac{1}{2} - 5 \cos \theta}$

(iii) $y = \frac{1}{\frac{1}{3} - 4 \sin (2\theta - 5)}$

(iv) $y = \frac{1}{3 + \frac{2}{5} \sin (5\theta - 7)}$

3. Find domain and range of the following:

(i) $y = 7 \cos 4x$

(ii) $y = \cos \frac{x}{3}$

(iii) $y = \sin \frac{2x}{3}$

(iv) $y = 7 \cot \frac{\pi}{2} x$

(v) $y = 4 \tan \pi x$

(vi) $y = \operatorname{cosec} 4x$

4. Check whether the following functions are odd or even?

(i) $y = \sin x + x \cos x$

(ii) $y = x^3 \sin x \cos x$

(iii) $y = \frac{x^2 \tan x}{x + \sin x}$

(iv) $y = x^3 \sin x \cos^2 x$

(v) $y = \frac{\sin^2 x}{x + \tan x}$

(vi) $y = \frac{\tan x - \sin x}{\sin^3 x}$

(vii) $y = \frac{\sec x}{x + \tan x}$

(viii) $y = x^2 \sin x - \cot x$

5. Draw the graph of each of the following functions:

(i) $y = 2 \sin x$

(ii) $y = 2 \cos 3x$

(iii) $y = 2 \tan 2x$

(iv) $y = \cos \frac{x}{2}$

(v) $y = 2 \sin 3x$

(vi) $y = 3 \cos x$

(vii) $y = \cos^2 x$

(viii) $y = \sin^2 x$

(ix) $y = \tan^2 x$

(x) $y = \sin \frac{x}{2}$

6. Find the periods of the following:

(i) $y = 6 \sec (2x - 3)$

(ii) $y = \cos (5x + 4)$

(iii) $y = \cot 4x + \sin \frac{5x}{2}$

(iv) $y = 7 \sin(3x + 3)$ (v) $y = 5 \sin(2x + 3)$ (vi) $y = 2 \tan 3x + 7 \cos x$

7. Draw the graphs of $y = \sin x$ and $y = \sin 2x$ in $[0, 2\pi]$ on the same scale.
8. Draw the graphs of $y = \cos x$ and $y = \cos 2x$ in $[0, 2\pi]$ on the same scale.
9. Solve graphically:
 - (i) $\sin x = \cos x$ (ii) $\cos x = x$
 - (iii) $\sin x = x$ (iv) $\tan x = x$
10. Alternating current cyclically reverses direction. The maximum voltage is about 180 volts when the standard frequency is 56 Hz (56 cycles per second). The voltage can be modeled by

$$V(t) = a \sin(k(t - d)) + c$$

Determine each of the following:

- a. Period of 56 Hz AC,
- b. The value of k ,
- c. The amplitude of the voltage function,
- d. Model the voltage with an appropriate transformed Sine function.

I have Learnt

- Finding the domain and range of the trigonometric functions
- Discussing even, and odd functions and the periodicity of trigonometric functions
- Finding the maximum and minimum value of a given function of the type:
 - $a + b \sin \theta$,
 - $a + b \cos \theta$,
 - $a + b \sin(c\theta + d)$,
 - $a + b \cos(c\theta + d)$,
 - The reciprocal of above, where a, b, c , and d are real numbers.
- Graphing and analyzing the trigonometric functions sine, cosine, and tangent to solve problems.
- Explaining the properties of graphs of $\sin \theta$, $\cos \theta$ and $\tan \theta$.
- Applying the concepts of trigonometric functions, identities, graphs, periodicity, even, odd functions, and extreme values to real-world problems such as (distance, elevation, and direction of tall structures, navigation and mapping, lengths of irregular shapes, graphs to visualize and predict patterns in data, frequency and periodic length of Ferris wheel, forces on a see-saw or lever, the ideal angle for solar panel placement)

Review Exercise

1. Mark the correct option in each of the following:
 - i. If $\cos \theta = -\frac{\sqrt{3}}{2}$ and the terminal arm of angle is in III quadrant. Then $\sin \theta = \dots$
 - a. $\frac{1}{2}$
 - b. $-\frac{1}{2}$
 - c. $\sqrt{3}$
 - d. $-\frac{2}{\sqrt{3}}$
 - ii. The exact value of the trigonometric function $\tan(-15\pi) = \dots$
 - a. 0
 - b. -1
 - c. 1
 - d. Undefined
 - iii. If $2\sin \theta + \frac{1}{2}\text{Cosec } \theta$ and $\theta = 45^\circ$, then the value of the given trigonometric identity is:
 - a. $\frac{1}{\sqrt{2}}$
 - b. $\frac{1}{3}$
 - c. $\frac{3}{\sqrt{2}}$
 - d. $\frac{\sqrt{2}}{3}$
 - iv. If $\sin(270^\circ + \theta) = x$ and the terminal side of an angle ' θ ' is in IV quadrant, then $x = \dots$
 - a. $\cos \theta$
 - b. $-\cos \theta$
 - c. $\sin \theta$
 - d. $-\sin \theta$
 - v. The trigonometric identity $\frac{\sin \alpha + \sin 2\alpha}{1 + \cos \alpha + \cos 2\alpha} = \dots$
 - a. $\sin \alpha$
 - b. $\cos \alpha$
 - c. $\tan \alpha$
 - d. $\cot \alpha$
 - vi. Express $2 \sin 3x \sin 7x$ as a sum or difference:
 - a. $\cos 4x - \cos 10x$
 - b. $\cos 10x - \cos 4x$
 - c. $\cos 4x + \cos 10x$
 - d. $\cos 10x + \cos 4x$
 - vii. Express $\sin 5x + \sin 7x$ as a product:
 - a. $2 \sin 6x \cos x$
 - b. $2 \sin x \cos 6x$
 - c. $2 \cos 7x \sin 5x$
 - d. $2 \cos 5x \sin 7x$
 - viii. The value of $\tan x \cdot \tan\left(\frac{\pi}{3} - x\right) \cdot \tan\left(\frac{\pi}{3} + x\right)$ is:
 - a. $2 \cot 3x$
 - b. $\cot 3x$
 - c. $3 \tan 3x$
 - d. $\tan 3x$
 - ix. If $\tan A = \frac{1}{7}$, and $\tan B = \frac{1}{3}$, Then $\cos 2A$ is equal to:
 - a. $\sin B$
 - b. $\sin 4B$
 - c. $\sin 3B$
 - d. $\cos 2B$
 - x. Whether the function $f(x) = \frac{\sin^3 x}{x^2 + \tan x}$ is:
 - a. even
 - b. odd
 - c. neither even nor odd
 - d. both even and odd
 - xi. The period of $\cos \frac{x}{5}$ is:
 - a. 10π
 - b. $\frac{2\pi}{5}$
 - c. 2π
 - d. 4π

xii. The trigonometric function $y = \operatorname{cosec} x$ meet at $x = \dots$

- a. 30° b. 60° c. 90° d. 120°

xiii. $2 \cos 5x \cdot \sin 3x = \dots$

- a. $\sin 8x - \sin 2x$ b. $\sin 8x + \sin 2x$ c. $\cos 8x + \cos 2x$ d. $\sin 4x - \sin x$

xiv. The trigonometric functions which are even and having period $= 2\pi$ are:

- a. $\sin x$ & $\cos x$ b. $\sec x$ & $\cos x$ c. $\sin x$ & $\operatorname{cosec} x$ d. $\tan x$ & $\cot x$

xv. If ' f ' is a periodic function and its period is π , then $f(\theta)$ could be equal to:

- a. $2 \cos x$ b. $2 \cos 3x$ c. $3 \cos 2x$ d. $\cos 4x$

xvi. If function $f(x) = \sin 8x$ is a periodic function and its period equals:

- a. π b. $\frac{\pi}{4}$ c. 2π d. $\frac{\pi}{2}$

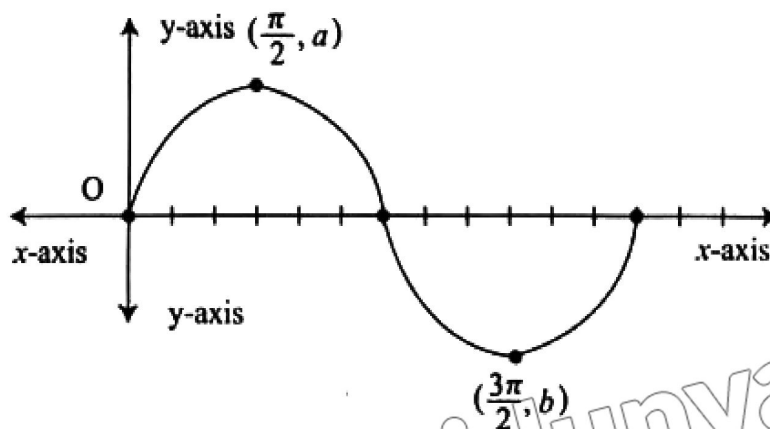
xvii. If the range of the function $f(\theta) = a \sin(2\theta) + b$, where $a > 0$, is $[3, 5]$, then $3a + 2b = \dots$

- a. 11 b. 14 c. 9 d. 5

xviii. The minimum value of the trigonometric function $f(\theta) = 17 \sin(4\theta)$ is:

- a. 4 b. -4 c. -17 d. 17

xix. If the given figure represent the curve $y = 3 \sin x$, then $|a| + |b| = \dots$



- a. 1 b. 2 c. 3 d. 6

xx. The maximum value of $7 \cos x + 24 \sin x$ is:

- a. 25 b. -25 c. 7 d. 24

2. If $\cos \theta - \sin \theta = \sqrt{2} \sin \theta$, then show that $\cos \theta + \sin \theta = \sqrt{2} \cos \theta$.

3. Verify the following Trigonometric identities:

(a) $\frac{\tan x - \cot x}{\sin x \cos x} = \sec^2 x - \operatorname{cosec}^2 x$

(b) $\frac{\sec^4 x - \tan^4 x}{\sec^2 x + \tan^2 x} = \sec^2 x - \tan^2 x$

(c) $\frac{\sin t}{1 - \cos t} - \frac{\sin t \cos t}{1 + \cos t} = \operatorname{cosec} t (1 + \cos^2 t)$

4. Prove that:

(i) $\frac{\tan(\alpha + \beta) - \tan \beta}{1 + \tan(\alpha + \beta) \tan \beta} = \tan \alpha$

(ii) $\frac{1 + \sin 2\theta - \cos 2\theta}{1 + \sin 2\theta + \cos 2\theta} = \tan \theta$

5. If $\cos \alpha + \cos \beta + \cos \gamma = 0$, then prove that:

$$\cos \alpha \cdot \cos \beta \cdot \cos \gamma = \frac{1}{12} (\cos 3\alpha + \cos 3\beta + \cos 3\gamma).$$

6. Prove that:

$$\frac{\cos 7\alpha - \cos 8\alpha}{1 + 2 \cos 5\alpha} = \frac{\sin 7\alpha - \sin 8\alpha}{1 + 2 \sin 5\alpha} = \cos 2\alpha - \cos 3\alpha$$

7. A Ferris wheel is 40 meters in diameter and boarded from a platform that is 4 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 16 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn.

a. Find the period, amplitude and vertical shift of $h(t)$.

b. Find a formula for the height function $h(t)$.

c. How high off the ground is a person after 5 minutes?

8. The 'h' (in meters) above the ground of a rider on a Ferris wheel, 't' (in seconds) after the rider begins is given:

$$h(t) = 10 \sin(3(t - 30)) + 12$$

Determine each of the following:

- The maximum and minimum heights of the rider above the ground.
- The height of the rider above the ground 30 seconds after start.
- The time required for the Ferris Wheel to complete one complete revolution.

9. The top of the flagpole sways back and forth in high winds. The top sways 8cm to the right (+ 8 cm) and 8 cm to the left (− 3 cm) of its resting position. It moves back and forth 260 every minute. The pole was momentarily at its rest position at $t = 0$, before it started moving to the right.

Find:

- the equation of the sinusoidal function that describes the distance the top of the pole is from resting position in terms of time elapsed.
- the domain and range correspond to the situation described.

10. Find the domain, range, and period of the following trigonometric functions:

(i) $7 \sin \frac{\pi}{4} x$

(ii) $2 \sin \frac{x}{3}$

(iii) $\frac{3}{2} \operatorname{cosec} x$

(iv) $5 \sin 3x$

(v) $\frac{1}{2} \sin \frac{2x}{3}$

(vi) $\frac{5}{3} \cos \frac{4x}{3}$

(vii) $3 \sin \pi x$

(viii) $7 \sin 5x$

(ix) $\frac{1}{3} \sec \frac{\pi}{4} x$

(x) $\frac{3}{2} \cot \frac{2\pi}{3} x$

(xi) $9 \cos (3x - 2)$

(xii) $8 - \cos 4x$

(xiii) $7 + 5 \sin (2x - \frac{\pi}{6})$

(xiv) $6 + 4 \cos (2x + \frac{\pi}{3})$

(xv) $\frac{1}{1 - \sin x}$