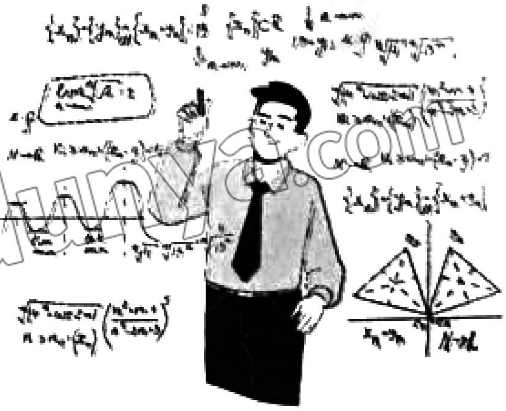
UNIT
07

MATHEMATICAL INDUCTION AND BINOMIAL THEOREM

After studying this unit, students will be able to:

- Describe a mathematical argument, identify the base case, induction of hypothesis and a precise conclusion.
- Apply the principle of mathematical induction to prove statements, identities, divisibility of numbers and summation formulae.
- Evaluate and justify conclusions, communicating a position clearly in an appropriate mathematical form in daily life.
- State and apply the Binomial Theorem to expand expressions of the form $(a+b)^n$ where n is a positive integer.
- Describe Binomial Theorem as expansion of binomial powers restricted to the set of natural numbers.
- Calculate binomial coefficients using Pascal's triangle.
- Expand using the binomial theorems, and use appropriate techniques to simplify the expression.
- Find an approximate value using binomial theorem.
- Use binomial theorem to find the remainder when a number to some large exponent is divided by a number.
- Use binomial theorem to find the last digit of a number, test the divisibility by a number and compare two large numbers.
- Apply concepts of Mathematical induction and binomial theorem to real world problems such as (puzzles, domino effects, Pascal's triangle, Economic forecasting, Rankings, Variable subletting).

The concept of mathematical induction was first utilized by the Italian scientist Francesco Maurolico in 1575. During the seventeenth century, both Pierre de Fermat and Blaise Pascal also employed this technique, with Fermat referring to it as the "method of infinite descent." In 1883, Augustus De Morgan, renowned for De Morgan's laws, provided a meticulous description of the process and named it mathematical induction.



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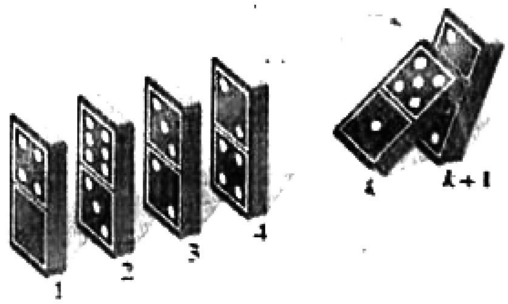
7.1 Mathematical Induction

To illustrate the idea of mathematical induction, envision an infinite sequence of dominoes arranged in a line, where if one domino falls backward, it causes the next one to fall backward as well. Now, suppose the first domino falls backward. What occurs next? . . . They all fall down. (Figure 7.1)



If the kth domino falls backward, it will also push the (k + 1)th domino backward.

To establish the connection between this visualization and the principle of mathematical induction, consider the sentence "The nth domino falls backward", denoted as P(n). It is known that for every k ≥ 1, if P(k) is true (the kth domino falls backward), then P(k + 1) is also true (the (k + 1)th domino falls backward). Additionally, it is given that P(1) is true (the first domino falls backward). Hence, according to the principle of mathematical induction, P(n) (the nth domino falls backward) is true for every integer n ≥ 1.

7.1.2 Principle of Mathematical Induction

Example: Use the method of mathematical induction to prove that

$$1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all positive integers $'n'$.

Solution:

Here the proposition $P(n)$ is:

$$1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Step 1: (Basis Step)

$P(1)$ is true; since

$$1.2 = \frac{1(1+1)(1+2)}{3}$$

$$\Rightarrow 2 = 2$$

> **Key Facts**
> The two steps are involved in the mathematical induction. First one is known as basis step and next one is known as inducive step.

Step 2: (Inductive Step)

$P(k+1)$ is true whenever $P(k)$ is true.

Let $P(n)$ is true for $n = k$. i.e.;

$$1.2 + 2.3 + 3.4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Now we prove that $P(k+1)$ is also true. For this we add $(k+1)(\overline{k+1}+1)$ on both sides.

$$1.2 + 2.3 + 3.4 + \cdots + k(k+1) + (k+1)(\overline{k+1}+1)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(\overline{k+1}+1)$$

$$1.2 + 2.3 + 3.4 + \cdots + k(k+1) + (k+1)(\overline{k+1}+1) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k+1)(k+2)\left[\frac{k}{3}+1\right] = (k+1)(k+2)\left[\frac{k+3}{3}\right]$$

$$= \frac{(k+1)(\overline{k+1}+1)(\overline{k+1}+2)}{3}$$

This shows that $P(k+1)$ is true. Thus, it is true for all positive integers.

Example: Use the method of mathematical induction to prove that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$$

for all positive integers 'n'.

Solution:

Here the proposition $P(n)$ is:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$$

Step 1: (Basis Step)

For $n = 1$; $P(1)$ is

$$\left(1-\frac{1}{2}\right) = \frac{1}{1+1}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{2}$$

This shows that $P(1)$ is true.

Step 2: (Inductive Step)

In this step we will prove that $P(k+1)$ is true whenever $P(k)$ is true.

Let it is true for $n = k$; i.e.;

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k+1}\right) = \frac{1}{k+1}$$

Now multiply both sides by $\left(1-\frac{1}{k+1+1}\right)$.

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k+1}\right)\left(1-\frac{1}{\overline{k+1}+1}\right) = \left(\frac{1}{k+1}\right)\left(1-\frac{1}{\overline{k+1}+1}\right)$$

$$= \left(\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right) = \left(\frac{1}{k+1}\right)\left(\frac{k+2-1}{k+2}\right) = \left(\frac{1}{k+1}\right)\left(\frac{k+1}{k+2}\right) = \frac{1}{k+2}$$

$$= \frac{1}{\overline{k+1}+1}$$

This shows that it is true for $n = k+1$ i.e.; $P(k+1)$ is true.

Thus, it is true for all positive integers n.

Example: Use the method of mathematical induction to show that $n^2 - 3n + 4$ is an even (i.e.; divisible by 2) for all positive integers n.

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Solution:

The proposition $P(n)$ $n^2 - 3n + 4$ is an even number for all positive integers.

Step 1: (Basis Step)

For $n = 1$; $P(1)$ is $1^2 - 3(1) + 4 = 2$ which is an even number. Thus $P(1)$ is true.

Step 2: (Inductive Step)

$P(k+1)$ is true when $P(k)$ is true.

Let $P(k)$ is true i.e.; $k^2 - 3k + 4$ is an even. Now $P(k+1)$ is:

$$\begin{aligned}(k+1)^2 - 3(k+1) + 4 &= k^2 + 2k + 1 - 3k - 3 + 4\\ &= (k^2 - 3k + 4) + (2k + 1 - 3)\\ &= (k^2 - 3k + 4) + (2k - 2)\\ &= (k^2 - 3k + 4) + 2(k - 1)\end{aligned}$$

Which is an even because it is sum of two even numbers $(k^2 - 3k + 4)$ and $2(k-1)$.

$\Rightarrow$ $P(k+1)$ is true. Thus, it is true for all positive integers n.

Example: Use the method of mathematical induction to show that $3^n > n^2$ for all positive integers n.

Solution:

The proposition $P(n)$ is $3^n > n^2$ for all positive integers n.

Step 1: (Basis Step)

For $n = 1$

$P(1)$ is $3^1 > 1^2 \Rightarrow 3 > 1$

$\Rightarrow P(1)$ is true.

Step 2: (Inductive Step)

$P(k+1)$ is true when $P(k)$ is true.

Let it is true for $n = k$. i.e.;

$$3^k > k^2$$

Now $3^{k+1} = 3 \times 3^k = 3^k + 3^k + 3^k > 3^k + 3^k$

$$\Rightarrow 3^{k+1} > k^2 + 3^k \qquad \because 3^k > k^2 \text{ is true for } n = k$$

Also $3^k > 2k + 1$ for $k > 1$

$\Rightarrow 3^{k+1} > k^2 + 2k + 1$

$\Rightarrow 3^{k+1} > (k+1)^2$

This shows that $P(k+1)$ is true. Thus, true for all positive integers n.

Example: Use the method of mathematical induction to show that

$4 + 4.6 + 4.6^2 + 4.6^3 + \cdots + 4.6^n = \frac{4(6^{n+1}-1)}{5}$ for all positive integers n.

Solution:

We have to prove the proposition P(n) that $4 + 4.6 + 4.6^2 + 4.6^3 + \cdots + 4.6^n = \frac{4(6^{n+1}-1)}{5}$ by mathematical induction.

Step 1: (Basis Step)

For $n = 0$

$$4=\frac{4(6^{0+1}-1)}{5}=4\times\frac{5}{5}=4$$

$\Rightarrow$P(0) is true

Step 2: (Inductive Step)

$P(k+1)$ is true when $P(k)$ is true.

Let it is true for $n=k$. i.e.;

$$4+4.6+4.6^2+4.6^3+\cdots+4.6^k=\frac{4(6^{k+1}-1)}{5} \qquad (1)$$

Now we have to show that proposition is true for n = k + 1.

Adding 4.6^{k+1}on both sides of equation (1), we get:

$$\begin{aligned}4+4.6+4.6^2+4.6^3+\cdots+4.6^k+4.6^{k+1}&=\frac{4(6^{k+1}-1)}{5}+4.6^{k+1}\\&=\frac{4.(6^{k+1}-1+5.6^{k+1})}{5}\\&=\frac{4.(6.6^{k+1}-1)}{5}=\frac{4.(6^{k+2}-1)}{5}\end{aligned}$$

This shows that $P(k+1)$ is true. Thus, true for all positive integers n.

Exercise 7.1

By the method of the mathematical induction prove the following when n is an integer.

1. $1+2+3+\cdots+n=\frac{n(n+1)}{2} \quad \forall n\geq 1$
2. $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6} \quad \forall n\geq 1$
3. $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4} \quad \forall n\geq 1$
4. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots+\frac{1}{(3n-2)(2n+1)}=\frac{n}{3n+1} \quad \forall n\geq 1$
5. $1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{n(4n^2-1)}{3} \quad \forall n\geq 1$
6. $4^3+4^4+4^5+\cdots+4^3=\frac{n(4^n-16)}{3} \quad \forall n\geq 3$
7. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \quad \forall n\geq 1$
8. $\frac{3}{1.2.2}+\frac{4}{2.3.2^2}+\frac{5}{3.4.2^3}+\cdots+\frac{n+2}{n(n+1).2^n}=1-\frac{1}{(n+1).2^n} \quad \forall n\geq 1$
9. $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\cdots+\frac{n+4}{n(n+1)(n+2)}=\frac{n(3n+7)}{2(n+1)(n+2)} \quad \forall n\geq 1$
10. $7+77+777+\cdots+777\ldots7_{n\ times}=\frac{7}{81}(10^{n+1}-9n-10) \quad \forall n\geq 1$
11. $1^3+3^3+5^3+\cdots+(2n+1)^3=(n+1)^2(2n^2+4n+1) \quad \forall n\geq 0$
12. $1.2^0+2.2^1+3.2^2+\cdots+n.2^{n-1}=(n-1).2^n+1 \quad \forall n\geq 1$

13. $1.1! + 2.2! + 3.3! + \cdots n.n! = (n+1)! - 1$ $\quad\forall n \geq 1$

14. $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ $\quad\forall n \geq 2$

15. $\left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\left(\frac{1}{5}\cdot\frac{1}{6}\right)\ldots\left(\frac{1}{2n+1}\cdot\frac{1}{2n+2}\right) = \frac{1}{(2n+2)!}$ $\quad\forall n \geq 0$

16. $1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n = \frac{1-(-2)^n}{3}$ $\quad\forall n \geq 1$

17. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ $\quad\forall n \geq 0$

18. $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n.2^{n-1}$ $\quad\forall n \geq 1$

19. $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$ $\quad\forall n \geq 0$

Prove the followings by mathematical induction.

20. $n^3 + 2n$ is divisible by 3 $\quad\forall n \geq 1$
21. 6 is a factor of $n(n^2+5)$ $\quad\forall n \geq 1$
22. $\frac{n(3n^4+5n^2+7)}{15}$ is a rational number.
23. $4^n + 15n - 1$ is divisible by 9 $\quad\forall n \geq 1$
24. $7^n - 2^n$ is divisible by 5 $\quad\forall n \geq 0$

Prove the followings inequalities by using the method of mathematical induction.

25. $2^n < (n+1)!$ $\quad\forall n \geq 2$
26. $5^n + 9 < 6^n$ $\quad\forall n \geq 2$
27. If $h > -1$ then $1 + nh \leq (1+h)^n$ $\quad\forall n \geq 0$
28. $\binom{2n}{n} < 2^{2n-2}$ $\quad\forall n \geq 5$
29. $\sqrt[n]{n} < 2 - \frac{1}{n}$ $\quad\forall n \geq 2$
30. $1 + 3n \leq 4^n$ $\quad\forall n \geq 0$
31. $n^3 > 2n + 1$ $\quad\forall n \geq 2$
32. $n! > n^2$ $\quad\forall n \geq 4$

7.2 Binomial Theorem

'Bi' mean two and 'nominal' mean terms. So, binomial mean an algebraic expression consisting of two terms. e.g., $(x+y)$, $\left(\frac{1}{x}-\frac{1}{2}\right)$, $\left(x^2+\frac{1}{x}\right)$ *etc* all are binomials.

Often, we need some positive integral powers of binomial like square, cube or even higher powers. Higher is the power the longer will be the expansion. To handle such problem we use binomial theorem. General form of binomial expression is a $(a+b)^n$ where n is a positive integer. We can expand the expression $(a+b)^n$ by using binomial theorem. Another way to expand $(a+b)^n$ is the use of Pascal's triangle.

7.2.1 Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

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This triangle of positive integers is known as Pascal's triangle. How the Pascal triangle is helpful in the binomial expansion; observe some positive integral powers of $(a+b)^n$.

for $n = 1$ $\quad (a+b)^1 = a+b$

for $n = 2$ $\quad (a+b)^2 = a^2 + 2ab + b^2$

for $n = 3$ $\quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

for $n = 4$ $\quad (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Observe that the binomial coefficients in the above expansion are.

for $n = 1$ $\quad$ 1 1

for $n = 2$ $\quad$ 1 2 1

for $n = 3$ $\quad$ 1 3 3 1

for $n = 4$ $\quad$ 1 4 6 4 1

Which are same as the first four rows of the Pascal's triangle.

In this way we can find the binomial coefficients from the Pascal triangle by considering its n^{th} row; for the expansion of $(a+b)^n$. Also from the above expansion note that, expansion starts with a^n and in each next term exponent of a is decreased by 1 and the exponent of b is increased by 1. The expansion ends with the term b^n.

Key Facts

> Expansion by using Pascal's triangle is convenient when n is a small positive integer.

Example: Expand $(1+2x)^6$ with the help of pascal's triangle.

Solution:

Here $a = 1$; $b = 2x$ and $n = 6$. For the binomial coefficients we need the 6^{th} row of the Pascal's triangle.

Here $a = 1$; $b = 2x$ and $n = 6$. For the binomial coefficients we need the 6^{th} row of the Pascal's triangle.

1st Row $\longrightarrow$ 1 1

2nd Row $\longrightarrow$ 1 2 1

3rd Row $\longrightarrow$ 1 3 3 1

4th Row $\longrightarrow$ 1 4 6 4 1

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5th Row $\longrightarrow$ 1 5 10 10 5 1
6th Row $\longrightarrow$ 1 6 15 20 15 6 1

Thus

$$\begin{aligned}(1+2x)^2 &= 1(1)^6 + 6(1)^5(2x) + 15(1)^4(2x)^2 + 20(1)^3(2x)^3 + 15(1)^2(2x)^4 + 6(1)^1(2x)^5 \\ &\quad + 1(2x)^6 \\ &= 1(1) + 6(2x) + 15(4x^2) + 20(8x^3) + 15(16x^4) + 6(32x^5) + 64x^6 \\ &= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\end{aligned}$$

Example: Expand $\left(2-\frac{1}{x}\right)^5$ with the help of Pascal's triangle.

Solution:

$$\left(2-\frac{1}{x}\right)^5 = \left[2+\left(-\frac{1}{x}\right)\right]^5$$

Here $a = 2$; $b = -\frac{1}{x}$ and $n = 5$. For binomial coefficients we need 5th row of the Pascal's triangle.

1st Row $\longrightarrow$ 1 1
2nd Row $\longrightarrow$ 1 2 1
3rd Row $\longrightarrow$ 1 3 3 1
4th Row $\longrightarrow$ 1 4 6 4 1
5th Row $\longrightarrow$ 1 5 10 10 5 1

Therefore

$$\begin{aligned}\left[2+\left(-\frac{1}{x}\right)\right]^5 &= 1(2)^5 + 5(2)^4\left(-\frac{1}{x}\right) + 10(2)^3\left(-\frac{1}{x}\right)^2 + 10(2)^2\left(-\frac{1}{x}\right)^3 + 5(2)^1\left(-\frac{1}{x}\right)^4 \\ &\quad + 1\left(-\frac{1}{x}\right)^5 \\ &= 1(32) + 5(16)\left(-\frac{1}{x}\right) + 10(8)\left(\frac{1}{x^2}\right) + 10(4)\left(-\frac{1}{x^3}\right) + 5(2)\left(\frac{1}{x^4}\right) + 1\left(-\frac{1}{x^5}\right) \\ &= 32 - \frac{80}{x} + \frac{80}{x^2} - \frac{40}{x^3} + \frac{10}{x^4} - \frac{1}{x^5}\end{aligned}$$

7.2.2 Binomial Theorem

Statement: If a and b are any two real numbers and n is a positive integer then

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$$

Proof:

We will prove this with the help of mathematical induction.

Step 1: (Basis Step)

For $n = 1$

$$(a+b)^1 = \binom{1}{0}a^1b^0 + \binom{1}{1}a^{1-1}b^1 = (1)a(1) + (1)(1)b = a + b$$

True for $n = 1$

Step 2: (Inductive Step)

Let it is true for $n = k$. i.e.;

$$(a+b)^k = \binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b^1 + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}a^0 b^k$$

Now we will prove that it is true for $n = k+1$. For this multiply the above equation by $a+b$ on both sides.

$$(a+b)(a+b)^k = (a+b)\left[\binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b^1 + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}a^0 b^k\right]$$

$$\begin{aligned}
\Rightarrow (a+b)^{k+1} &= a\left[\binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b^1 + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}a^0 b^k\right] \\
&\quad + b\left[\binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b^1 + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}a^0 b^k\right] \\
&= \left[\binom{k}{0}a^{k+1}b^0 + \binom{k}{1}a^k b^1 + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}a^0 b^k\right] \\
&\quad + \left[\binom{k}{0}a^k b^1 + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}a^0 b^{k+1}\right]
\end{aligned}$$

By collecting the like terms, we have,

$$(a+b)^{k+1} = \binom{k}{0}a^{k+1}b^0 + \left[\binom{k}{1} + \binom{k}{0}\right]a^k b + \left[\binom{k}{2} + \binom{k}{1}\right]a^{k-1}b^2 + \cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}a^0 b^{k+1}$$

Since,

$$\binom{k}{0} = 1 = \binom{k+1}{0}$$

$$\binom{k}{k} = 1 = \binom{k+1}{k+1}$$

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r} \qquad for\ 0 \le r \le k$$

Thus

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1}b^0 + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k+1-1}ab^k + \binom{k+1}{k+1}a^0 b^{k+1}$$

This shows that it is true for $n = k+1$; hence it is true for all positive integer n.

Some Properties of Binomial Expansion

1. The number of terms in the expansion of $(a+b)^n$ is one more than the index n.
2. The sum of exponents of a and b in each term of the expansion of $(a+b)^n$ is n.
3. The coefficients of the terms equidistant from the beginning and the end are same.
4. If n is even then there will be odd number of terms in the expansion of $(a+b)^n$. So the middle term in this expansion is the $\left(\frac{n}{2}+1\right)^{th} = \left(\frac{n+2}{2}\right)^{th}$ term.

5. If n is odd then there will be even number of terms in the expansion of $(a+b)^n$. So there will be two middle terms in the expansion; these are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms of the expansion.
6. In the expansion of $(a+b)^n$, exponent of a is n and the exponent of b is zero in the first term. In each next term exponent of a is decreased by 1 and the exponent of b is increased by 1. In the last term exponent a becomes zero and the exponent of b is reached to n.
7. Any particular $(r+1)^{th}$ term from beginning also known as general term in the expansion of $(a+b)^n$ is given by
$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$
8. A term which is at r^{th} position from the end in the expansion of $(a+b)^n$ is at $(n+2-r)^{th}$ position from the beginning.

Example: Expand $(x-y)^5$ using binomial theorem.

Solution:

$$(x-y)^5 = [x+(-y)]^5$$
$$= \binom{5}{0}x^5(-y)^0 + \binom{5}{1}x^4(-y)^1 + \binom{5}{2}x^3(-y)^2 + \binom{5}{3}x^2(-y)^3 + \binom{5}{4}x^1(-y)^4 + \binom{5}{5}x^0(-y)^5 \quad (1)$$

Now the binomial coefficients are;

$$\binom{5}{0} = \frac{5!}{0!\,(5-0)!} = \frac{5!}{1\times 5!} = 1$$
$$\binom{5}{1} = \frac{5!}{1!\,(5-1)!} = \frac{5!}{1\times 4!} = \frac{5\times 4!}{4!} = 5$$
$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = \frac{5!}{2!\times 3!} = \frac{5\times 4\times 3!}{2\times 1\times 3!} = 10$$
$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\times 2!} = \frac{5\times 4\times 3!}{3!\times 2\times 1} = 10$$
$$\binom{5}{4} = \frac{5!}{4!\,(5-4)!} = \frac{5!}{4!\times 1!} = \frac{5\times 4!}{4!\times 1} = 5$$
$$\binom{5}{5} = \frac{5!}{0!\,(5-0)!} = \frac{5!}{1\times 5!} = 1$$

Substituting values in equation (1)

$$= (1)x^5(-y)^0 + (5)x^4(-y)^1 + (10)x^3(-y)^2 + (10)x^2(-y)^3 + (5)x^1(-y)^4 + (1)x^0(-y)^5$$
$$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

Example: Find the constant term in the expansion of $\left(x+\frac{2}{x}\right)^{10}$.

Solution:

The constant term in the expansion is independent of 'x'.

Here $a = x$; $b = \frac{2}{x}$ and $n = 10$

The general term of binomial expansion is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Substituting the values

$$T_{r+1} = \binom{10}{r} x^{10-r} \left(\frac{2}{x}\right)^r = \binom{10}{r} x^{10-r} \frac{2^r}{x^r}$$

$$= \binom{10}{r} x^{10-2r} 2^r \qquad (1)$$

Term will be independent of x if the exponent of x is zero i.e.; $10 - 2r = 0 \Rightarrow 2r = 10$

$$\Rightarrow r = 5$$

Putting value of r in equation (1), we have

$$T_{5+1} = \binom{10}{5} x^0 2^5 = \frac{10!}{5!\,(10-5)!}.(1)(32) = \frac{10!}{5!\,5!}.32 = \frac{10.9.8.7.6.5!}{5.4.3.2.1.5!}(32)$$

$$T_6 = 8064$$

Example: Find the 3rd term from the end in the expansion of $\left(2 - \frac{5}{\sqrt{x}}\right)^5$.

Solution: Here $a = 2$, $b = \frac{-5}{\sqrt{x}}$, $n = 5$

3rd term from the end is at $5 + 2 - 3 = 4^{th}$ from the beginning.

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

For fourth term, $r = 3$

Substituting the values;

$$T_{3+1} = \binom{5}{3} 2^{5-3} \left(\frac{-5}{\sqrt{x}}\right)^3 = \frac{5!}{3!\,(5-3)!} 2^2 \left(\frac{-125}{x^{\frac{3}{2}}}\right) = \frac{5!}{3!\,2!}(4)\left(\frac{-125}{x^{\frac{3}{2}}}\right) = \frac{5.4.3!}{3!.2.1}\left(\frac{-500}{x^{\frac{3}{2}}}\right)$$

$$T_4 = 10\left(-500x^{-\frac{3}{2}}\right) = -5000x^{\frac{-3}{2}}$$

Example:

Find the remainder when 7^{101} is divided by 25.

Solution:

$$\frac{7^{101}}{25} = \frac{7.7^{100}}{25} = \frac{7.(7^2)^{50}}{25} = \frac{7(49)^{50}}{25} = \frac{7}{25}(50-1)^{50}$$

$$= \frac{7}{25}\left[\binom{50}{0}(50)^{50}(-1)^0 + \binom{50}{1}(50)^{49}(-1)^1 + \binom{50}{2}(50)^{48}(-1)^2 + \cdots + \binom{50}{49}(50)^1(-1)^{49} + \binom{50}{50}(50)^0(-1)^{50}\right]$$

$$= \frac{7}{25}\left[\left\{(50)^{50}(-1)^0 + \binom{50}{1}(50)^{49}(-1)^1 + \binom{50}{2}(50)^{48}(-1)^2 + \cdots + \binom{50}{49}(50)^1(-1)^{49}\right\} + \binom{50}{50}\right]$$

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$$= \frac{7}{25}\left[(50)^{50}(-1)^0 + \binom{50}{1}(50)^{49}(-1)^1 + \binom{50}{2}(50)^{48}(-1)^2 + \cdots + \binom{50}{49}(50)^1(-1)^{49}\right] + \frac{7}{25}$$

Thus, the remainder is 7.

Illustration

If we divide 25 by 7 the remainder is 4. We may write

$$\frac{25}{7} = 3 + \frac{4}{7}$$

Numerator of the fractional part is the remainder.

Example: The fourth term in the expansion of $\left(ax + \frac{1}{x}\right)^n$ is $\frac{5}{2}$

Find the values of a and n.

Solution:

General term of the binomial expansion is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

The fourth term in the expansion of $\left(ax + \frac{1}{x}\right)^n$ is

$$\Rightarrow T_{3+1} = \binom{n}{3}(ax)^{n-3}\left(\frac{1}{x}\right)^3 = \binom{n}{3} a^{n-3} x^{n-6}$$

Given that fourth term is $\frac{5}{2}$; thus

$$\binom{n}{3} a^{n-3} x^{n-6} = \frac{5}{2} \qquad (1)$$

Now right side of (1) is independent of x, this needs exponent of x to be zero. i.e.; $n - 6 = 0$

$$\Rightarrow n = 6$$

Putting value of n in equation (1), we get

$$\binom{n}{3} a^{n-3} x^0 = \frac{5}{2} \quad \Rightarrow \frac{6!}{3!(6-3)!} a^3 = \frac{5}{2} \qquad \Rightarrow \frac{6.5.4.3!}{3.2.13!} a^3 = \frac{5}{2}$$

$$\Rightarrow 20a^3 = \frac{5}{2} \quad \Rightarrow a^3 = \frac{5}{40}$$

$$\Rightarrow \; a^3 = \frac{1}{8} = \left(\frac{1}{2}\right)^3$$

$$\Rightarrow a = \frac{1}{2}$$

7.2.3 Use of Pascal's Triangle and Binomial Theorem in Real World Problems

Pascal triangle and binomial theorem are used in the real-world problems such as cryptography, calculating the number of matches played in a tournament where n teams are playing, calculating the possible number of protein structure and DNA sequence.

Example: Use pascal triangle to find the possible number of heads when three coins are tossed simultaneously.

Solution:

When three coins are tossed together the following are the possible results.

HHH HHT HTH HTT THH THT TTH TTT

0 Heads = one result

1 Head = 3 results

2 Heads = 3 results

3 Heads = one result

By pascal triangle we have

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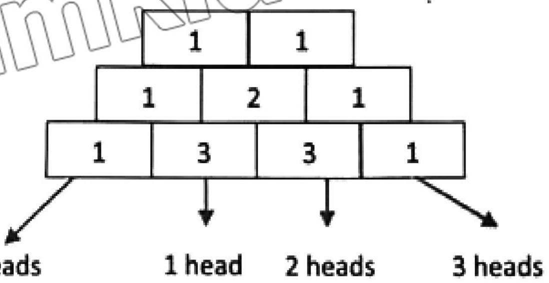
By pascal triangle we have



Exercise 7.2

1. Expand the following with the help of Pascal's triangle.

   (i) $\left(2\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5$ (ii) $\left(\frac{3}{x}+\frac{y}{2}\right)^6$ (iii) $\left(2-x^{3/2}\right)^7$ (iv) $\left(\frac{x^2}{y^2}-\sqrt{\frac{y}{x}}\right)^5$

2. Expand the followings by using binomial theorem.

   (i) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^5$ (ii) $(-x+y^{-1})^6$ (iii) $(3u-1)^7$ (iv) $\left(a\sqrt{2}+b\sqrt{3}\right)^5$

   (v) $(1+2x-y)^4$ (vi) $\left(\frac{1}{x}+\frac{2}{y}+\frac{3}{z}\right)^4$

3. Expand and simplify.

   (i) $(1+10x)^4+(1-10x)^4$ (ii) $\left(2-\frac{3}{x}\right)(1+4x^2)$

   (iii) $(1+2x+2x^2)(1-x)^5$ (iv) $(.99)^3+(1.01)^4$

   (v) $\left(\frac{2}{x}-\frac{x}{4}\right)^4+\left(\frac{2}{x}+\frac{x}{4}\right)^4$ (vi) $\left(a^2+\sqrt{a^2-1}\right)^4-\left(a^2-\sqrt{a^2-1}\right)^4$

4. Find the coefficient of the 8^{th} term in the expansion of $\left(x^2+\frac{y}{2}\right)^{10}$.

5. Find the middle term in the expansion of the following.

   (i) $\left(3x^2-\frac{1}{2x}\right)^{10}$ (ii) $\left(2x^2-\frac{1}{5x}\right)^{11}$

   (iii) $\left(\frac{a}{\sqrt{x}}+\sqrt{x}\right)^8$ (iv) $\left(a-\frac{3}{x^2}\right)^{12}$

6. Find the specified term in the following expansions.

   (i) Term involving b^6 in the expansion of $\left(\frac{a^2}{2}+2b^2\right)^{10}$

   (ii) Term involving q^8 in the expansion of $\left(\frac{p^2}{2}+6q^2\right)^{12}$

   (iii) Term involving x^4y^3 in the expansion of $(3x^4-y)^4$

   (iv) Term involving y^8x^3 in the expansion of $(y^4-3x)^5$

7. Find the term independent of x in the expansions of the following.

   (i) $\left(2x^2-\frac{1}{x}\right)^{12}$ (ii) $\left(\sqrt{x}+\frac{1}{3x^2}\right)^{10}$

8. Find the r^{th} term from the end in the expansion of $(a+b)^n$ where $0\le r\le n$.

9. Prove that sum of all the binomial coefficients in the expansion of $(a+b)^n$ is 2^n; hence or otherwise prove that sum of odd coefficients is 2^{n-1}.
10. The sum of coefficients of first three terms in the expansion of $\left(a-\frac{3}{a^2}\right)^n$ is 559. Find the term involving a^3 in the expansion.
11. If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ term in the expansion of $(1+a)^{34}$ are equal; then find the value of r.
12. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2m}$ are in A.P., show that $2m^2-9m+7=0$.
13. If coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio $6:33:110$, then find the value of n and the position of terms.
14. Prove that $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$
15. Prove that $\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots+\frac{(-1)^n}{n+1}\binom{n}{n}=\frac{1}{n+1}$
16. Prove that $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{2^2}\binom{n}{2}+\cdots+\frac{1}{2^n}\binom{n}{n}=\left(\frac{3}{2}\right)^n$
17. Prove that $\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}$
18. Use pascals triangle to find the number of heads when six coins are tossed simultaneously.
19. If 7 coins are tossed how many times 5 heads will appear.
20. If a coin is tossed 8 times how many times 3 tails will appear.

7.3 Binomial Series

7.3.1 Expansion of $(1+x)^n$ when n is Positive Integer

Since the index n of $(1+x)^n$ is a positive integer; so by using the binomial theorem we have,

$$(1+x)^n=\binom{n}{0}(1)^n x^0+\binom{n}{1}(1)^{n-1}x^1+\binom{n}{2}(1)^{n-2}x^2+\cdots+\binom{n}{n}(1)^0x^n$$

$$=\frac{n!}{0!\,(n-0)!}(1)(1)+\frac{n!}{1!\,(n-1)!}(1)x+\frac{n!}{2!\,(n-2)!}(1)x^2+\cdots+\frac{n!}{n!\,0!}x^n$$

$$=\frac{n!}{n!}(1)+\frac{n(n-1)!}{(n-1)!}x+\frac{n(n-1)(n-2)!}{2!\,(n-2)!}x^2+\cdots++\frac{n!}{n!}x^n$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots+x^n$$

The series on the right is terminating and has $(n+1)$ number of terms.

7.3.2 Expansion of $(1+x)^n$ when n is not Positive integers or Fractional Number

When n is not positive or fractional number then expansion of $(1+x)^n$ is non-terminating. i.e.;

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots+\infty$$

The above series will be convergent if $|x| < 1$ or $-1 < x < 1$.
Convergent means series has a finite sum otherwise series will be divergent. We will focus only those expansions of $(1+x)^n$ which are convergent.

The series $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$ is known as binomial series. The general term of the binomial series is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$$

Exercise 7.3

1. Expand the followings upto four terms and also find the values of x for which the series is convergent.

(i). $\left(1-\sqrt{x}\right)^{-3}$ (ii). $\left(3+\frac{2}{x}\right)^{-\frac{1}{3}}$ (iii). $\left(\frac{5}{2}-\frac{3}{x^2}\right)^{\frac{1}{2}}$

(iv). $\frac{3+x}{3-x}$ (v). $\frac{1-2x}{\sqrt{3+\frac{x}{2}}}$ (vi) $\frac{\sqrt{x+1}}{\sqrt{1-x}}$

2. Approximate the value upto four places of decimal.

(i). $\sqrt[6]{65}$ (ii). $\sqrt{\frac{5}{3}}$ (iii). $(1.03)^{\frac{1}{3}}$ (iv) $(.95)^{\frac{2}{7}}$

3. Find the term involving x^{14} in the product of $(1+x^2)\left(2+\sqrt{3}x^3\right)^{-\frac{1}{2}}$.

4. If x is so small that its square and higher powers may be neglected then prove that:

(i) $\frac{\sqrt{2+x}(1-x)^{3/2}}{3+x} \approx \frac{\sqrt{2}}{3}\left(1-\frac{19}{12}x\right)$ (ii) $\frac{\left(1+\frac{2}{3}x\right)^{-5}+\sqrt{4+2x}}{(4+x)^{3/1}} \approx \frac{1}{8}\left(3-\frac{95}{24}x\right)$

(iii) $\frac{\sqrt{9-x}+\left(1+\frac{3}{4}x\right)^{-5}}{2+x} \approx 2-\frac{67}{24}x$

5. If x is so small that its cube and higher powers may be neglected then show that:

(i) $\frac{(1+x)^{3/2}-(1+x^2)^3}{\sqrt{1-x}} \approx -\frac{3x^2}{8}$ (ii) $\frac{(4+x)^{-2}+(1-2x)^{-5}}{(1+2x)^2} \approx \frac{3}{2}+\frac{15}{4}x+\frac{1251}{32}x^2$

6. If x is so large that $\left(\frac{1}{x}\right)^2$ and higher powers may be neglected then shoe that:

$$\sqrt{x^2+25}-\sqrt{x^2+9} \approx \frac{8}{x}$$

7. Find the term involving x^n while simplifying $\frac{(2+x)^2}{(1+x)^3}$.

8. Identify as binomial series and find the sum of the following:

(i) $1-\frac{3}{7}+4.\left(\frac{3}{7}\right)^2-8.\left(\frac{3}{7}\right)^3+\cdots$

(ii) $1-\frac{1}{15}+\frac{4}{2!.15^2}-\frac{4.7}{3!.15^3}+\cdots$

(iii) $1-\frac{2.4}{5}+\frac{3.4^2}{5^2}-\frac{4.4^3}{5^3}+\cdots$

(iv) $3-\frac{3}{18}-\frac{3}{2!.18^2}-\frac{3^2}{3!.18^3}-\cdots$

9. (i) If $y=\frac{3}{2^2.1!}+\frac{3}{2^4.2!}-\frac{3}{2^6.3!}+\cdots$, then show that $8y^2+16y-19=0$.

(ii) If $\frac{2}{y}=\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\cdots$ then show that $y^2-2y-2=0$.

10. (i) If x is very nearly equal to 1 then show that $\frac{ax^b-bx^a}{x^b-x^a}\approx\frac{1}{1-x}$.

(ii) If p and q are approximately equal; then prove that $\frac{q+2p}{p+2q}\approx\left(\frac{p}{q}\right)^{1/3}$.

Hence approximate the value of $\left(\frac{2.01}{2}\right)^{1/3}$.

Applications of Binomial Theorem

The binomial theorem has a wide range of applications in Mathematics, like finding the remainder, finding the digits of a number, etc. The most common binomial theorem applications are as follows:

Finding the Last or Unit Place digit of an exponential number

Consider the table given below in which numbers are written in first column while their exponents are written in the first row and it is showing only the unit place digits.

Powers → Numbers ↓	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	2	4	8	6	2	4	8	6	2
3	3	9	7	1	3	9	7	1	3
4	4	6	4	6	4	6	4	6	4
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6
7	7	9	3	1	7	9	3	1	7
8	8	4	2	6	8	4	2	6	8
9	9	1	9	1	9	1	9	1	9

From the table, the unit place digits for 2 are as $2^1=2, 2^2=4, 2^3=8, 2^4=16$ and $2^5=32$. Since the unit digit is same as that of the number 2 therefore the cyclicity of 2 is 5 – 1 =4. In the same manner we can find the cyclicity of other numbers. The cyclicity of 3, 7 and 8 is also 4. For 1, 5 and 6 the unit digit remains the same for all the exponents and for 4 the unit place digit is either 4 or 6 whereas for 9 it is 9 or 1 only. Hence, we can say that if the exponent is of the form 4n, 4n+1, 4n+2 or 4n+3 then we can easily find the value of the unit place digits of all the numbers.

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Example: Find the unit digit of: (i) 17^{203} (ii) 29^{26} (iii) 36^{307}

Solution:

(i) 17^{203}

Now 203 can be written as:

$203 = 4 \times 50 + 3$

Since, the remainder is 3 so, $7^3 = 343$

Hence the unit place digit of 17^{203} is 3.

(ii) 29^{26}

As in case of 9 from the table we can see that there are only two values i.e.; 9 and 1.

So, write 26 as $4 \times 6 + 2$. Hence remainder is 2 that is $9^2 = 81$. Unit place digit of 9^{26} is 1.

(iii) 36^{307}

As the unit place digit is 6 which always remains 6 at unit place, so 36^{307} has 6 at unit place.

Finding Remainder Using Binomial Theorem

Example: Find the remainder when 7^{103} is divided by 25.

Solution:

$$\frac{7^{103}}{25} = \frac{7(49)^{51}}{25} = \frac{7(50-1)^{51}}{25}$$

$$= \frac{7(25k-1)}{25} = \frac{175k - 25 + 25 - 7)}{25}$$

$$= \frac{25(7k-1)+18}{25}$$

$\therefore$ The remainder = 18

Example: If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then find k.

Solution:

$$\frac{2^{403}}{15} = \frac{2^3\,(2^4)^{100}}{15}$$

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$= \frac{8}{15}(16)^{100} = \frac{8}{15}(15+1)^{100} = \frac{8}{15}(15\lambda + 1) = 8\lambda + \frac{8}{15}$

$\because 8\lambda$ is an integer, fractional part $= \frac{8}{15}$

So, $k = 8$

Example Find the unit digit of (i) 17^{203} (ii) 29^{26} (iii) 36^{307}

Finding Digits of a Number

Example: Find the last two digits of the number $(13)^{10}$.

Solution:

$(13)^{10} = (169)^5 = (170-1)^5$

$= {}^5C_0(170)^5 - {}^5C_1(170)^4 + {}^5C_2(170)^3 - {}^5C_3(170)^2 + {}^5C_4(170) - {}^5C_5$

$= {}^5C_0(170)^5 - {}^5C_1(170)^4 + {}^5C_2(170)^3 - {}^5C_3(170)^2 + {}^5C_4(170) - 1$

A multiple of $100 + 5(170) - 1 = 100k + 849$

$\therefore$ The last two digits are 49.

Relation between Two Numbers

Example: Which one is greater $99^{50} + 100^{50}$ or 101^{50}?

Solution:

101^{50} can be written as:

$101^{50} = (100+1)^{50} = 100^{50} + 50 \,.\, 100^{49} + 25 \,.\, 49 \,.\, 100^{48} + \ldots$

99^{50} can be written as:

$\Rightarrow 99^{50} = (100-1)^{50} = 100^{50} - 50 \,.\, 100^{49} + 25 \,.\, 49 \,.\, 100^{48} - \ldots.$

Now, $101^{50} - 99^{50} = 2[50 \,.\, 100^{49} + 25(49)(16)\,100^{47} + \ldots]$

$= 100^{50} + 50 \,.\, 49 \,.\, 16 \,.\, 100^{47} + \ldots > 100^{50}$

$\therefore 101^{50} - 99^{50} > 100^{50}$

$\Rightarrow 101^{50} > 100^{50} + 99^{50}$

Divisibility Test

Example: Show that $11^9 + 9^{11}$ is divisible by 10.

Solution:

$11^9 + 9^{11} = (10+1)^9 + (10-1)^{11}$

$= [{}^9C_0 \times 10^9 + {}^9C_1 \times 10^8 + \ldots + {}^9C_9] + [{}^{11}C_0 \times 10^{11} - {}^{11}C_1 \times 10^{10} + \ldots + {}^{11}C_{11}]$

$= {}^9C_0 \times 10^9 + {}^9C_1 \times 10^8 + \ldots + {}^9C_8 \times 10 + 1 + 10^{11} - {}^{11}C_1 \times 10^{10} + \ldots + {}^{11}C_{10} \times 10 - 1$

$= 10[{}^9C_0 \times 10^8 + {}^9C_1 \times 10^7 + \ldots + {}^9C_8 + {}^{11}C_0 \times 10^{10} - {}^{11}C_1 \times 10^9 + \ldots + {}^{11}C_{10}]$

$= 10k$, which is divisible by 10.

Exercise 7.4

1. Find the unit place digits in 27^{304}, 108^{33}, 54^{203} and 503^{43}.
2. Find the remainder when:
   a. 9^{205} is divided by 31. b. 8^{205} is divided by 48.
3. If fractional part of number $\frac{2^{510}}{31}$ is $\frac{k}{31}$, then find the value of k.
4. Find the last one or two digits of the number where applicable.
   a. 15^8 b. 37^7 c. 29^{10}
5. Which of the following is a larger number?
   a. $98^{50} + 100^{50}$ or 102^{50} b. $47^{30} + 50^{30}$ or 53^{30}
6. Show that $12^{15} + 8^{15}$ is divisible by 10.
7. Show that $22^{25} + 18^{25}$ is divisible by 20.
8. Use binomial theorem to find the remainder when 5^{103} is divided by 13.
9. What is the remainder when 17^{1717} is divided by 9.
10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.
11. Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.
12. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.
13. Show that $6^{n+3} - 8n - 6$ is divisible by 6.

I have Learnt

- Describing a mathematical argument and identifying the base case, induction of hypothesis and a precise conclusion.
- Applying the principle of mathematical induction to prove statements, identities, divisibility of numbers and summation formulae.
- Evaluating and justifying conclusions, communicating a position clearly in an appropriate mathematical form in daily life.
- Stating and applying the Binomial Theorem to expand expressions of the form $(a+b)^n$ where n is a positive integer.
- Describing Binomial Theorem as expansion of binomial powers restricted to the set of natural numbers.
- Calculating binomial coefficients using Pascal's triangle.
- Expanding the binomial theorems, and using appropriate techniques to simplify the expression.
- Finding an approximate value using binomial theorem.
- Using binomial theorem to find the remainder when a number to some large exponent is divided by a number.
- Using binomial theorem to find the last digit of a number, testing the divisibility by a number and compare two large numbers.
- Applying concepts of Mathematical induction and binomial theorem to real world problems such as (puzzles, domino effects, Pascal's triangle, Economic forecasting, Rankings, Variable subletting).

Review Exercise

1. Select the correct option:

(i) Mathematical induction is used to check a proposition for all n where n is a/an:
(a) real number (b) rational number (c) integer (d) positive integer

(ii) A mathematical statement which is true for all positive integers is also true for all:
(a) negative integers (b) positive numbers (c) whole numbers (d) none

(iii) If n is even positive integer then the middle term in the expansion of $(a+b)^n$ is:
(a) $\left(\frac{n}{2}\right)^{th}$ term (b) $\left(\frac{n+1}{2}\right)^{th}$ term (c) $\left(\frac{n-1}{2}\right)^{th}$ term (d) $\left(\frac{n}{2}+1\right)^{th}$ term

(iv) In the expansion of $(a+b)^{20}$ a term is at the 11^{th} position. Its position from the end is:
(a) 9^{th} (b) 10^{th} (c) 11^{th} (d) 12^{th}

(v) The coefficient of the 3^{rd} last term in the expansion of $(1+x)^{300}$ is:
(a) 277 (b) 44850 (c) 303 (d) 4305600

(vi) $\binom{11}{0}+\binom{11}{2}+\binom{11}{4}+\cdots+\binom{11}{10}$ is equal to
(a) 2^{11} (b) 2^{12} (c) 2^{10} (d) $2^{11}-1$

(vii) If the third term in the expansion of $(1+x)^p$ is $-\frac{1}{8}x^2$ then the value of p is:

(a) 2 (b) $^1/_2$ (c) 4 (d) 3

(viii) The coefficient of x^n in the expansion of $(1+x+x^2+\cdots)^{-n}$ where n is a even number:

(a) 1 (b) -1 (c) n (d) $-n+1$

(ix) The greatest coefficient in the expansion of $(1+x)^{10}$ is:

(a) 2^x (b) $\binom{10}{5}$ (c) $\binom{10}{6}$ (d) 2^{10}

(x) Binomial series $(2+3x)^{-1/2}$ is valid when:

(a) $|x| \leq 1$ (b) $|x| < 1$ (c) $|x| < \frac{2}{3}$ (d) $|x| < \frac{3}{2}$

2. Using principle of mathematical induction prove that for all positive integers$n$:

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}++\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

3. The ratio of coefficients of three consecutive terms in the binomial expansion of $(1+x)^n$ is 2 : 15 : 70. Find the average of the three coefficients.

4. Show that the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ does not contain any term involving $\frac{1}{x}$.

5. If α and β are nearly equal then show that $\left(\frac{3\beta}{5\alpha-2\beta}\right)^{-1/3} \approx \frac{\alpha}{\alpha+2\beta}+\frac{\alpha+\beta}{3\beta}$

6. If $^{22}C_r$ is the largest coefficient in the expansion of $(1+x)^{22}$ then find $^{13}C_r$.

7. Use binomial theorem to prove that 6^n-5^n leaves a remainder 1, when divided by 5.

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