

UNIT 02

MATRICES AND DETERMINANTS

After studying this unit, students will be able to:

- Apply matrix operations (addition/subtraction and multiplication of (matrices) with real and complex entries.
- Evaluate determinants of 3×3 matrix by using cofactors and properties of determinants.
- Use row operations to find the inverse and the rank of a matrix.
- Explain a consistent and inconsistent system of linear equations and demonstrate through examples
- Solve a system of 3 by 3 nonhomogeneous linear equations by using matrix inversion method and Cramer's Rule.
- Solve a system of three homogeneous linear equations in three unknowns using the Gaussian elimination method.
- Apply concepts of matrices to real world problems such as (graphic design, data encryption, seismic analysis, cryptography, transformation of geometric shapes, social network analysis).

A very common use of matrices in daily life is encryption. We use them to scramble data for security purpose and to encode and decode this data. There is a key that helps encode and decode data which is generated by matrices. The screen of any electronic device, like smart phone or LED TV, screen is essentially a pixel matrix. When we rotate the phone and it is in landscape form. The matrix is actually rotated using the transpose. When we touch the screen of a cell phone at some specific position; the position is calculated by matrix properties.

In mathematics we use matrices to solve the system of linear equations. Matrices are also used frequently almost in all sciences.

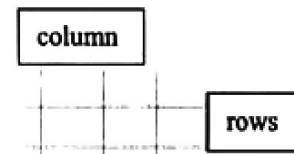
In 19th century the term matrix was introduced by English mathematician James Sylvester. Then after taking the idea of matrices from Sylvester, Arthur Cayley developed the algebra of matrices and published two papers in 1850s. On system of linear equations matrices was applied by Cayley's where they are still useful.



2.1 Matrices

A matrix is an array of numbers arranged in horizontal and vertical lines enclosed within square brackets. Matrices are usually denoted with capital letters.

The horizontal lines are known as rows of the matrix and vertical lines are known as columns of the matrix. e.g.;



$$A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 8 \end{bmatrix}$$

Each number in the matrix is called an element or entry of the matrix. Every element in the matrix has definite position which can be specified by the number of rows first and then number of column where it exists. In the above matrix position of element '8' is determined where second row and third column meet each other. In general, an element in the i th row and the j th column is denoted by a_{ij} and the matrix A generally is written as $A = [a_{ij}]$.

2.1.1 Order of a Matrix

How many rows and columns are there in a matrix is known as order of the matrix. If a matrix A has m number of rows and n number of columns then the order of the matrix is $m \times n$ or m -by- n . We always write number of rows first then number of columns.

If we multiply m by n ; it gives us the total number of elements in the matrix. e.g.; if there are 3 rows and 2 columns in a matrix A then its order is 3×2 ; often we write $A_{3 \times 2}$. The product of 3 and 2 is 6; so, there are six elements in matrix A .

Equality of Matrices

Any two matrices are said to be equal if both have same order and same corresponding elements.

Consider matrices $A = \begin{bmatrix} 3 & 5 \\ 2 & 1 \\ 7 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 2 & 1 \\ 7 & 9 \end{bmatrix}$.

Here both matrices A and B are of same order 3×2 and also have same corresponding elements. Thus they are equal. We write $A = B$.

2.1.2 Types of Matrices

Row Matrix or Row Vector

If there is only one row in a matrix then the matrix is called row matrix or row vector. e.g.;

$$A = [1 \ 3 \ 6 \ 2]_{1 \times 4}; \quad B = [2 \ 5]_{1 \times 2}; \quad C = [5]_{1 \times 1}$$

$D = [3 \ 0 \ 1 \ 9 \ 2]_{1 \times 5}$ are all row matrices.

In general, if a row matrix A having n number of columns is

$$[a_{11} \ a_{12} \ a_{13} \ \dots \ a_{1n}]_{1 \times n}$$

Column Matrix or Column Vector

If there is only one column in a matrix then the matrix is called column matrix or column vector.

$$\text{e.g. } A = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}_{4 \times 1} \quad B = \begin{bmatrix} 0 \\ 9 \\ 5 \end{bmatrix}_{3 \times 1} \quad C = \begin{bmatrix} 2 \\ 9 \end{bmatrix}_{2 \times 1} \quad D = [6]_{1 \times 1} \text{ are column matrices.}$$

In general, a column matrix with m number of rows is

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}_{m \times 1}$$

Square Matrix

A matrix which has equal number of rows and columns is called a square matrix. i.e.; if a matrix has n number of rows and n number of columns then it is called square matrix and its order is $n \times n$.

e.g.; $\begin{bmatrix} 3 & 1 \\ -5 & 7 \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & -2 \\ 2 & -3 & 5 \end{bmatrix}_{3 \times 3}$, $[3]_{1 \times 1}$ are square matrices.

Rectangular Matrix

If the number of rows are not equal to the number of columns in a matrix then the matrix is called a rectangular matrix. i.e.; if a matrix has m number of rows and n number of columns and $m \neq n$ then the matrix is a rectangular matrix. e.g.;

$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & -1 \end{bmatrix}_{2 \times 3}$, $\begin{bmatrix} 1 & 6 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}_{3 \times 2}$, $[1 \ -2 \ 3]_{1 \times 3}$ are rectangular matrices.

Zero or Null Matrix

If all the entries (elements) of a matrix are zero then the matrix is called null or zero matrix. A zero matrix is usually denoted by $O_{m \times n}$. e.g.;

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[0 \ 0 \ 0 \ 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $[0]$ are zero matrices.

Diagonal of a Matrix

Consider a square matrix A of order 3×3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Secondary diagonal

Main diagonal

Then elements a_{11}, a_{22}, a_{33} with same subscripts form the main diagonal or principal diagonal of the matrix and the elements a_{13}, a_{22}, a_{31} in which 1st script is increased by 1 and 2nd is decreased by 1 form the secondary diagonal of matrix. In general, for a square matrix A of order $n \times n$ primary and secondary diagonals are shown as under:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2(n-1)} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3(n-1)} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

Secondary diagonal

Main diagonal

The elements of the main diagonal are $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ and the elements of the secondary diagonal are $a_{1n}, a_{2n-1}, \dots, a_{n1}$.

Diagonal Matrix

A square matrix in which all the elements except the main diagonal are zero and the main diagonal has at least one non zero element is called a diagonal matrix.

If $A = [a_{ij}]_{n \times n}$ is a square matrix of order n then it is called a diagonal matrix if $a_{ij} = 0$ when $i \neq j$ and $a_{ij} \neq 0$ for atleast one $i = j$ where $i = 1, 2, 3, \dots, n$ and $j = 1, 2, 3, \dots, n$.

e.g.; $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$; $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $[3]$ are diagonal matrices.

Scalar Matrix

A diagonal matrix in which all the diagonal elements are same but not zero is called a scalar matrix.

i.e.; if $A = [a_{ij}]_{n \times n}$ and $\begin{cases} a_{ij} = 0 \text{ for } i \neq j \\ a_{ij} = k \text{ for } i = j \end{cases}$ where k is a non-zero scalar.

e.g.; $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ where $k \neq 0$; $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ are scalar matrices.

Identity or Unit Matrix

A scalar matrix in which all the diagonal elements are equal to 1 is known as an identity matrix.

An identity matrix is usually denoted by $I_{n \times n}$ or simply I . For an identity matrix

$$I = [a_{ij}]; \begin{cases} a_{ij} = 0 \text{ for } i \neq j \\ a_{ij} = 1 \text{ for } i = j \end{cases}$$

e.g.; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$, $[1]_{1 \times 1}$ are identity matrices.

Upper Triangular Matrix

A square matrix in which all the elements lying below the main diagonal are zero; is called an

upper triangular matrix. i.e.; if $A = [a_{ij}]_{n \times n}$ and $a_{ij} = 0$ where $i > j$; $\begin{pmatrix} i = 1, 2, 3, \dots, n \\ j = 1, 2, 3, \dots, n \end{pmatrix}$ then A is an upper triangular matrix.

e.g.; $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$ are upper triangular matrices.

Lower Triangular Matrix

A square matrix in which all the elements lying above the main diagonal are zero; is called a

lower triangular matrix. i.e.; if $A = [a_{ij}]_{n \times n}$ and $a_{ij} = 0$ where $i < j$; $\begin{pmatrix} i = 1, 2, 3, \dots, n \\ j = 1, 2, 3, \dots, n \end{pmatrix}$ then A is a lower triangular matrix.

e.g.; $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix}$ are lower triangular matrices.

Triangular Matrix

A square matrix which is either upper triangular or lower triangular is called a triangular matrix.

Key Facts



- Sum, difference or product of upper (lower) triangular matrices is again upper (lower) matrix.
- Diagonal matrix is both upper and lower triangular matrices.

Transpose of a Matrix

If A is any matrix of order $m \times n$ then the matrix which is obtained by interchanging rows with columns of the matrix is called transpose of the matrix and is denoted by A^t . Note that the order of the A^t is $n \times m$.

e.g.; if $A = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 3 \end{bmatrix}_{2 \times 3}$ then

$$A^t = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 6 & 3 \end{bmatrix}_{3 \times 2}$$



- Key Facts
- If A is square matrix, then order of A and A^t is same.
 - Transpose of lower triangular matrix is an upper triangular matrix and vice versa.

Symmetric Matrix

For a square matrix A if $A = A^t$ then A is called a symmetric matrix, e.g.; if

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & -4 \\ 5 & -4 & 3 \end{bmatrix}, \text{ then } A^t = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & -4 \\ 5 & -4 & 3 \end{bmatrix}$$

Since $A = A^t$, so A is a symmetric matrix. Observe that in symmetric matrix $a_{ij} = a_{ji} \forall i \neq j$.

Skew Symmetric Matrix

A square matrix A is called skew symmetric if $A = -A^t$.

$$\text{e.g.; if } A = \begin{bmatrix} 0 & 2 & -6 \\ -2 & 0 & 5 \\ 6 & -5 & 0 \end{bmatrix}$$

$$\text{Then } A^t = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & -5 \\ -6 & 5 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 2 & -6 \\ -2 & 0 & 5 \\ 6 & -5 & 0 \end{bmatrix} = -A$$

So, A is skew symmetric.

Note that in a skew-symmetric matrix $a_{ij} = -a_{ji} \forall i \neq j$ and $a_{ii} = 0 \forall i = j$.

Exercise 2.1

1. Find the order of the following matrices.

$$(i) A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad (ii) B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \quad (iii) C = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$$

$$(iv) D = [2 \ 1 \ 6 \ 8] \quad (v) E = [3] \quad (vi) F = \begin{bmatrix} 3 & 6 \\ 9 & 2 \end{bmatrix}$$

2. Identify the following matrices as square matrix, rectangular matrix, row matrix or column matrix.

$$(i) A = \begin{bmatrix} 3 & 6 & 2 \\ 2 & 1 & 9 \end{bmatrix} \quad (ii) B = \begin{bmatrix} 1 & 1 \\ 3 & 6 \\ 2 & 6 \end{bmatrix} \quad (iii) C = \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}$$

$$(iv) D = \begin{bmatrix} 1 & 6 & 9 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad (v) E = [2 \ 0 \ 1] \quad (vi) F = [16]$$

3. Identify the diagonal matrix, scalar matrix, identity matrix, lower triangular matrix, upper triangular matrix.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 6 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix};$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad F = \begin{bmatrix} \sqrt{3} & 1 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix};$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

4. Find the transpose of the following matrices and identify which one of them are symmetric and which are skew-symmetric.

$$A = \begin{bmatrix} 2 & 0 \\ \sqrt{5} & 6 \\ 1 & 9 \end{bmatrix}; \quad B = [1 \ 6 \ 2 \ 0]; \quad C = \begin{bmatrix} 2 & 6 \\ 9 & 2 \end{bmatrix};$$

$$D = \begin{bmatrix} 0 & 1 & 9 \\ -1 & 0 & 5 \\ -9 & -5 & 0 \end{bmatrix}; \quad E = \begin{bmatrix} 3 & -6 & 9 \\ -6 & 2 & 0 \\ 9 & 0 & 0 \end{bmatrix}; \quad F = \begin{bmatrix} 9 & 0 & 1 \\ 0 & 6 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

2.2 Algebra of Matrices

2.2.1 Scalar Multiplication

If k is a non-zero scalar and $A = [a_{ij}]_{m \times n}$ is a matrix of order $m \times n$, then the product of matrix A and scalar k is denoted by the matrix kA , the matrix obtained by multiplying the scalar with each of the elements of the matrix A .

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$, then

$$kA = k \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & ka_{23} & \dots & ka_{2n} \\ ka_{31} & ka_{32} & ka_{33} & \dots & ka_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & ka_{m3} & \dots & ka_{mn} \end{bmatrix}$$

In particular if $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -3 & 5 \end{bmatrix}$ then

$$\begin{aligned} 2A &= 2 \begin{bmatrix} 3 & 2 & 1 \\ 4 & -3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 3 & 2 \times 2 & 2 \times 1 \\ 2 \times 4 & 2 \times -3 & 2 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 & 2 \\ 8 & -6 & 10 \end{bmatrix} \end{aligned}$$



Order of matrix A and kA is same.

Addition of Matrices

In general, we cannot add any two matrices. Only those matrices are conformable for addition which have the same order.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are any two matrices of same order $m \times n$ then $A + B$ is also a matrix of order $m \times n$ in which each of its elements is the sum of corresponding elements of A and B. If we assume that $A + B = C$ where $C = [c_{ij}]_{m \times n}$ then $c_{ij} = a_{ij} + b_{ij} \quad \forall i, j \in \mathbb{N}$

Subtraction of Matrices

Like addition of matrices, we can subtract two matrices which have same order.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are any two matrices of same order $m \times n$ then $A - B$ is also a matrix of order $m \times n$ in which each of its element is the difference of the corresponding elements of A and B. If we assume that $A - B = C$ where $C = [c_{ij}]_{m \times n}$ then $c_{ij} = a_{ij} - b_{ij} \quad \forall i, j \in \mathbb{N}$

Example: Find $A + B$ and $A - B$ where $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 6 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}$

Solution:

$$A + B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 + (-2) & 0 + 6 \\ 2 + 0 & 1 + 0 \\ 6 + 2 & 5 + 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 1 \\ 8 & 6 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 6 & 5 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 - (-2) & 0 - 6 \\ 2 - 0 & 1 - 0 \\ 6 - 2 & 5 - 1 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 2 & 1 \\ 4 & 4 \end{bmatrix}$$

Multiplication of Matrices

Two matrices A and B are conformable for multiplication if number of columns of A is equal to number of rows of B . If $A = [a_{ij}]_{m \times n}$ is a matrix of order $m \times n$ and $B = [b_{ij}]_{n \times p}$ is a matrix of order $n \times p$ then the order of AB is $m \times p$.

Assume that $AB = [c_{ij}]_{m \times p}$, where c_{ij} is the sum of the elements obtained by multiplying the corresponding elements of the i th row of a matrix A with corresponding elements of the j th column of matrix B . For $AB = C$, we have:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}; B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2j} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix}$$

$\swarrow$ i th row $\searrow$ j th column

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1j} & \dots & c_{1p} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2j} & \dots & c_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nj} & \dots & c_{np} \end{bmatrix}$$

$\rightarrow c_{ij} = (a_{i1})(b_{1j}) + (a_{i2})(b_{2j}) + \dots + (a_{in})(b_{nj})$
 where $i = 1, 2, 3, \dots, m$
 $j = 1, 2, 3, \dots, p$

Example: Find the product AB for the given matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 0 \end{bmatrix}_{3 \times 2}$; $B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 4 \end{bmatrix}_{2 \times 3}$

Solution:

Matrices A and B are conformable for the product of AB , since the number of columns of A and the number of rows of B is the same.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} (1)(3) + (3)(2) & (1)(1) + (3)(5) & (1)(0) + (3)(4) \\ (2)(3) + (1)(2) & (2)(1) + (1)(5) & (2)(0) + (1)(4) \\ (6)(3) + (0)(2) & (6)(1) + (0)(5) & (6)(0) + (0)(4) \end{bmatrix} \\ &= \begin{bmatrix} 3 + 6 & 1 + 15 & 0 + 12 \\ 6 + 3 & 2 + 5 & 0 + 4 \\ 18 + 0 & 6 + 0 & 0 + 0 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 16 & 12 \\ 9 & 7 & 4 \\ 18 & 6 & 0 \end{bmatrix} \end{aligned}$$



Key Facts
If two matrices A and B are conformable for the product AB , then it is not necessary that they are conformable for the product BA .

Note that order of $AB = 3 \times 2 \times 3 = 3 \times 3$

Example: Find the product AB for the given matrices.

$$A = \begin{bmatrix} 2i & 1 \\ -i & 3i \end{bmatrix}; \quad B = \begin{bmatrix} 3 & -i & i \\ 2i & 0 & -2i \end{bmatrix}$$

Solution:

The number of columns of A and the number of rows of B is same. So they are conformable for the product AB . Now

$$\begin{aligned} AB &= \begin{bmatrix} 2i & 1 \\ -i & 3i \end{bmatrix} \begin{bmatrix} 3 & -i & i \\ 2i & 0 & -2i \end{bmatrix} \\ &= \begin{bmatrix} (2i)(3) + 1(2i) & (2i)(-i) + (1)(0) & (2i)(i) + (1)(-2i) \\ (-i)(3) + (3i)(2i) & (-i)(-i) + (3i)(0) & (-i)(i) + (3i)(-2i) \end{bmatrix} \\ &= \begin{bmatrix} 6i + 2i & -2i^2 + 0 & 2i^2 - 2i \\ -3i + 6i^2 & i^2 + 0 & -i^2 - 6i^2 \end{bmatrix} \\ &= \begin{bmatrix} 8i & -2(-1) & 2(-1) - 2i \\ -3i + 6(-1) & -1 & -(-1) - 6(-1) \end{bmatrix} \\ &= \begin{bmatrix} 8i & 2 & -2 - 2i \\ -3i - 6 & -1 & 7 \end{bmatrix} \end{aligned}$$

Order of $AB = 2 \times 2 \times 3 = 2 \times 3$

2.2.2 Commutative Property of Matrices w. r. t. Addition

Any two matrices which are conformable for addition holds commutative property w. r. t. addition. Consider the two matrices $A = [a_{ij}]_{2 \times 3}$ and $B = [b_{ij}]_{2 \times 3}$; so,

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \\ A + B &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} & b_{13} + a_{13} \\ b_{21} + a_{21} & b_{22} + a_{22} & b_{23} + a_{23} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = B + A \end{aligned}$$

Example: Verify the commutative property of addition for the given matrices:

$$A = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & 4 \\ -7 & 3 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} A + B &= \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & 4 \\ -7 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3+0 & 1+(-1) & 6+3 \\ 2+1 & 1+2 & 3+4 \\ 0+(-7) & 2+3 & 1+1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 9 \\ 3 & 3 & 7 \\ -7 & 5 & 2 \end{bmatrix} \quad (1) \end{aligned}$$

And

$$\begin{aligned}
 B + A &= \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & 4 \\ -7 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0+3 & (-1)+1 & 3+6 \\ 1+2 & 2+1 & 4+3 \\ (-7)+0 & 3+2 & 1+1 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 0 & 9 \\ 3 & 3 & 7 \\ -7 & 5 & 2 \end{bmatrix} \quad (2)
 \end{aligned}$$

From (1) and (2) we have $A + B = B + A$, i.e.; commutative property holds w. r. t addition.

Commutative Property of Matrices w. r. t. Multiplication

In general, the commutative property w. r. t. multiplication for matrices do not hold. i.e.; $AB \neq BA$.

Examples: For the matrices $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$; show that $AB \neq BA$.

Solution:

$$\begin{aligned}
 AB &= \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} (1)(0) + (2)(2) + (1)(1) & (1)(1) + (2)(3) + (1)(4) \\ (3)(0) + (1)(2) + (6)(1) & (3)(1) + (1)(3) + (6)(4) \end{bmatrix} \\
 &= \begin{bmatrix} 0+4+1 & 1+6+4 \\ 0+2+6 & 3+3+24 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 8 & 30 \end{bmatrix} \\
 BA &= \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 6 \end{bmatrix} \\
 &= \begin{bmatrix} (0)(1) + (1)(3) & (0)(2) + (1)(1) & (0)(1) + (1)(6) \\ (2)(1) + (3)(3) & (2)(2) + (3)(1) & (2)(1) + (3)(6) \\ (1)(1) + (4)(3) & (1)(2) + (4)(1) & (1)(1) + (4)(6) \end{bmatrix} \\
 &= \begin{bmatrix} 0+3 & 0+1 & 0+6 \\ 2+9 & 4+3 & 2+18 \\ 1+12 & 2+4 & 1+24 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 6 \\ 11 & 7 & 20 \\ 13 & 6 & 25 \end{bmatrix}
 \end{aligned}$$

Clearly $AB \neq BA$.

2.2.3 Verification of $(AB)^t = B^t A^t$

Consider the two matrices A and B which are conformable for the product AB .

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 2 & 1 \end{bmatrix}_{3 \times 2}; \quad B = \begin{bmatrix} 0 & 1 & 6 \\ 2 & 1 & 0 \end{bmatrix}_{2 \times 3} \text{ then}$$

$$\begin{aligned}
 AB &= \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 6 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} (1)(0) + (2)(2) & (1)(1) + (2)(1) & (1)(6) + (2)(0) \\ (5)(0) + (6)(2) & (5)(1) + (6)(1) & (5)(6) + (6)(0) \\ (2)(0) + (1)(2) & (2)(1) + (1)(1) & (2)(6) + (1)(0) \end{bmatrix} \\
 &= \begin{bmatrix} 0+4 & 1+2 & 6+0 \\ 0+12 & 5+6 & 30+0 \\ 0+2 & 2+1 & 12+0 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 6 \\ 12 & 11 & 30 \\ 2 & 3 & 12 \end{bmatrix}
 \end{aligned}$$

$$\Rightarrow (AB)^t = \begin{bmatrix} 4 & 3 & 6 \\ 12 & 11 & 30 \\ 2 & 3 & 12 \end{bmatrix}^t = \begin{bmatrix} 4 & 12 & 2 \\ 3 & 11 & 3 \\ 6 & 30 & 12 \end{bmatrix} \quad (1)$$

Now

$$A^t = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 6 & 1 \end{bmatrix} \text{ and } B^t = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 6 & 0 \end{bmatrix}$$

$$\begin{aligned} B^t A^t &= \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 2 & 6 & 1 \end{bmatrix} = \begin{bmatrix} (0)(1) + (2)(2) & (0)(5) + (2)(6) & (0)(2) + (2)(1) \\ (1)(1) + (1)(2) & (1)(5) + (1)(6) & (1)(2) + (1)(1) \\ (6)(1) + (0)(2) & (6)(5) + (0)(6) & (6)(2) + (0)(1) \end{bmatrix} \\ &= \begin{bmatrix} 0+4 & 0+12 & 0+2 \\ 1+2 & 5+6 & 2+1 \\ 6+0 & 30+0 & 12+0 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 2 \\ 3 & 11 & 3 \\ 6 & 30 & 12 \end{bmatrix} \quad (2) \end{aligned}$$

From equation (1) and (2), we have $(AB)^t = B^t A^t$.

Example: Show that for the two matrices A and B which are conformable for addition:

$$(A + B)^t = A^t + B^t$$

Solution:

Consider any two matrices A and B of the same order.

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 2 & 6 \end{bmatrix} \text{ then}$$

$$A + B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+5 \\ e+2 & f+6 \end{bmatrix}$$

$$\Rightarrow (A + B)^t = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+5 \\ e+2 & f+6 \end{bmatrix}^t = \begin{bmatrix} a+1 & c+3 & e+2 \\ b+2 & d+5 & f+6 \end{bmatrix} \quad (1)$$

Now

$$A^t = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \text{ and } B^t = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 6 \end{bmatrix}$$

$$\begin{aligned} A^t + B^t &= \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} + \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} a+1 & c+3 & e+2 \\ b+2 & d+5 & f+6 \end{bmatrix} \quad (2) \end{aligned}$$

From (1) and (2) we have,

$$(A + B)^t = A^t + B^t$$

Example: Any square matrix can be written as the sum of two square matrices such that one of them is symmetric and the other is skew-symmetric.

Solution:

Consider any square matrix A . Let we can write it as sum of two square matrices P and Q where P is symmetric and Q is skew-symmetric. i.e.;

$$A = P + Q; P^t = P \text{ \& } Q^t = -Q \quad (1)$$

$$\Rightarrow A^t = (P + Q)^t = P^t + Q^t = P + (-Q)$$

$$\Rightarrow A^t = P - Q \quad (2)$$

Key Facts
For any two matrices A and B which are conformable for addition
 $(A + B)^t = A^t + B^t$. In general
 $(A_1 + A_2 + \dots + A_n)^t = A_1^t + A_2^t + \dots + A_n^t$

Adding equation (1) and equation (2), we get:

$$A + A^t = 2P \Rightarrow P = \frac{1}{2}(A + A^t)$$

Now subtracting equation (2) from equation (1), we have:

$$A - A^t = 2Q \Rightarrow Q = \frac{1}{2}(A - A^t)$$

Observe that

$$P^t = \left[\frac{1}{2}(A + A^t) \right]^t = \frac{1}{2}(A^t + (A^t)^t) = \frac{1}{2}(A^t + A) = \frac{1}{2}(A + A^t) = P$$

So, P is symmetric.

$$Q^t = \left[\frac{1}{2}(A - A^t) \right]^t = \frac{1}{2}(A^t - (A^t)^t) = \frac{1}{2}(A^t - A) = -\frac{1}{2}(A - A^t) = -Q$$

So, Q is skew-symmetric.

Example: Write the matrix $A = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 0 \\ -1 & 2 & 1 \end{bmatrix}$ as a sum of two matrices where one is symmetric and the other is skew-symmetric.

Solution:

Let $A = P + Q$ where P is symmetric and Q is skew-symmetric.

$$\text{So } P = \frac{1}{2}(A + A^t) = \frac{1}{2} \left(\begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 0 \\ -1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 2 \\ 5 & 0 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 6 & 3 & 4 \\ 3 & 12 & 2 \\ 4 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & \frac{3}{2} & 2 \\ \frac{3}{2} & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

And

$$\begin{aligned} Q &= \frac{1}{2}(A - A^t) = \frac{1}{2} \left(\begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 0 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 2 \\ 5 & 0 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & -2 \\ -6 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\frac{1}{2} & 3 \\ \frac{1}{2} & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} \end{aligned}$$

Thus $A = P + Q$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 0 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & \frac{3}{2} & 2 \\ \frac{3}{2} & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & 3 \\ \frac{1}{2} & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

Exercise 2.2

1. Construct a matrix $A = [a_{ij}]$ of order 2×2 for which:

(i) $a_{ij} = \frac{i+3j}{2}$ (ii) $a_{ij} = \frac{i \times j}{2}$ (iii) $a_{ij} = \frac{i}{j}$ (iv) $a_{ij} = \frac{2i-3j}{3}$

2. Construct a matrix $B = [b_{ij}]$ of order 3×3 for which:

(i) $b_{ij} = \frac{i^2-j}{3}$ (ii) $b_{ij} = \frac{i^2-j^2}{2i}$ (iii) $b_{ij} = \frac{2}{2i+j}$ (iv) $b_{ij} = \frac{i^2+j^2}{i+j}$

3. If $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 6 & 1 \\ -1 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix}$ then find a matrix C such that:

$$A + B + C = O$$

4. (i) Find $A \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) Find $X \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} X = \begin{bmatrix} 7/2 & 11 & 2 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

- (iii) If $A = \begin{bmatrix} 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 14 \end{bmatrix}$ then find a non-zero matrix C such that $AC = BC$.

- (iv) $\begin{bmatrix} xy & 4 \\ 0 & x+y \end{bmatrix} = \begin{bmatrix} 8 & z \\ t & 6 \end{bmatrix}$ then find the values of z , t and $x^2 + y^2$.

- (v) If $A = \begin{bmatrix} 3 & 4 \\ 7 & 6 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find α and β so that $A^2 + \alpha I = \beta A$.

- (vi) Find the values of x if $\begin{bmatrix} x & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = 0$.

5. If $X = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then prove that $X^2 - 4X - 5I = O$.

6. If $A = \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix}$ then find α and β such that, $A^2 + \alpha I = \beta A$.

7. If $A = \begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix}$ then

- (i) Prove that for all positive integers n , $A^n = \begin{bmatrix} x^n & 0 \\ y(x^n-1) & 1 \end{bmatrix}$.

- (ii) If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then prove that for all positive integers n ,

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

8. Consider any two particular matrices A and B of your choice of order 2×3 and 3×2 respectively and show that $(AB)^t = B^t A^t$.

9. Consider any two particular matrices A and B of your choice of order 3×3 and show that $(A+B)^t = A^t + B^t$.

10. If A and B are two matrices such that $AB = B$ and $BA = A$. Find $A^2 + B^2$.

11. If $A = [a_{ij}]$ is a matrix of order 3×3 and $a_{ij} = i^2 - j^2$. Check whether A is symmetric or skew-symmetric.

12. For any square matrix A ; prove that $(A^n)^t = (A^t)^n$.

13. Find the matrices X and Y such that $2X - Y = \begin{bmatrix} 1 & 6 & -3 \\ 2 & 1 & 7 \end{bmatrix}$ and $X + 3Y = \begin{bmatrix} 4 & 3 & 2 \\ 1 & -3 & 0 \end{bmatrix}$.

2.3 Determinants

Not all but every square matrix is associated with some number (real or complex). This number is called the determinant of the matrix.

If A is any square matrix then its determinant is denoted by $\det(A)$ or $|A|$.

Corresponding to the square matrix A of order n ,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The determinant of A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$



More than one square matrix can have same value of determinant.

Key Facts

For our convenience we consider the determinants of the square matrices of order up to 3×3 .

2.3.1 Determinant of Matrix of Order 2×2

Consider a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of Matrix of Order 3×3

Consider a matrix A of order 3×3 i.e.; $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

The associated determinant is $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

To find the value of this determinant; we express the above determinant into the sum or difference of determinants of order 2. This process of finding the value of the determinant is called expansion of the determinant.

We can expand a determinant from any row or any column. Since in a determinant of order 3; there are three rows namely R_1, R_2, R_3 and three columns C_1, C_2, C_3 , so we can expand the determinant in six different ways; but the value of determinant will remain the same in each case.

If we expand the above given determinant from 1st row i.e.; from R_1 then

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

This can be generalized for determinants of the square matrices of higher order.

Minor of an Element of a Square Matrix

Let we have any square matrix A of order n , i.e.; $A = [a_{ij}]_{n \times n}$; then the minor of the element a_{ij} of matrix is a determinant of the matrix of order $(n-1) \times (n-1)$ obtained by neglecting the i th row and j th column of the matrix A . Minor of a_{ij} is denoted by M_{ij} . For example; consider a matrix A of order 3×3 .

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minor of the element a_{21} is M_{21} where $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$ is the determinant obtained by neglecting 2nd row and 1st column of the matrix A . Likewise we can find all the minors of elements of the matrix A .

Cofactor of an Element of a Square Matrix

For any square matrix A of order $n \times n$, the cofactor of an element a_{ij} of matrix A is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$ e.g.; If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then cofactor of the element a_{21} is:

$$\begin{aligned} A_{21} &= (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (-1)(a_{12}a_{33} - a_{13}a_{32}) = -a_{12}a_{33} + a_{13}a_{32} \\ &= a_{13}a_{32} - a_{12}a_{33} \end{aligned}$$

Example: If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 6 \\ 3 & 0 & -4 \end{bmatrix}$ then find M_{12} , M_{23} and A_{12} and A_{23} .

Solution:

$$M_{12} = \begin{vmatrix} -1 & 6 \\ 3 & -4 \end{vmatrix} = (-1)(-4) - (3)(6) = 4 - 18 = -14$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} = (1)(0) - (3)(3) = 0 - 9 = -9$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (-14) = (-1)(-14) = 14$$

$$A_{23} = (-1)^{2+3} M_{23} = (-1)(-9) = 9$$

2.3.2 Evaluation of the Determinant of a Square Matrix Using Cofactors

Consider a square matrix A of order 3×3 .

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

If we expand it from first row then:

$$\begin{aligned}
 |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
 &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\
 &= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13} \\
 &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}
 \end{aligned}$$

If we expand the determinant from first column then:

$$\begin{aligned}
 |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\
 &= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \\
 &= a_{11}(-1)^{1+1}M_{11} + a_{21}(-1)^{2+1}M_{21} + a_{31}(-1)^{3+1}M_{31} \\
 |A| &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}
 \end{aligned}$$

From the above discussion it is clear that, $|A|$ can be evaluated by adding the product of elements with corresponding cofactors of any row or column of the matrix.

Example: If $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ -1 & 3 & 0 \end{bmatrix}$; then find $|A|$ using cofactors.

Solution:

First, we find cofactors of any one of the row or column of the given matrix. Let us find the cofactors of C_3 . The elements of C_3 are a_{13} , a_{23} and a_{33} . In this case $a_{13} = 6$, $a_{23} = 2$ and $a_{33} = 0$. Now we find their corresponding cofactors.

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} = (-1)^4(0 - (-1)) = 1(1) = 1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = (-1)^5(3 - (-2)) = (-1)(5) = -5$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)^6(1 - 0) = (1)(1) = 1$$

$$\begin{aligned}
 \therefore |A| &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \\
 &= 6(1) + 2(-5) + 0(1) = 6 - 10 + 0 = -4
 \end{aligned}$$

2.3.3 Singular and Non-Singular Matrices

Any square matrix A is called **singular** if $|A| = 0$.

If $|A| \neq 0$ then it is called **non-singular** matrix.

For example, for the matrices $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 12 \\ \frac{1}{2} & 2 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1)(2) - (3)(4) = 2 - 12 = -10 \neq 0$$

$$|B| = \begin{vmatrix} 3 & 12 \\ \frac{1}{2} & 2 \end{vmatrix} = (3)(2) - 12\left(\frac{1}{2}\right) = 6 - 6 = 0$$

Thus A is non-singular matrix and B is singular matrix.

2.3.4 Adjoint of a Square Matrix

Consider any square matrix A of order n :

$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ then the adjoint of A is written as $adj(A)$ and is the matrix

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

If the order of the matrix A is 3×3 . i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } adj(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Multiplicative Inverse of a Square Matrix

Two square matrices of same order n are said to be the multiplicative inverses of each other if their product is I_n (identity matrix of order n).

Only non-singular matrices have their multiplicative inverses.

If A is a non-singular matrix then its multiplicative inverse is denoted by A^{-1} and

$$AA^{-1} = A^{-1}A = I$$

2.3.5 Adjoint Method to Find the Inverse of a Non-Singular Matrix

If A is a non-singular square matrix i.e.; $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|} adj(A)$

Obviously if A is a singular then $|A| = 0$, then $A^{-1} = \frac{1}{|A|} adj(A)$ will not exist.

Example: If $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & 4 \end{bmatrix}$ then find A^{-1} by adjoint method.

Solution:

$$\text{Since } |A| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & 4 \end{vmatrix}$$

$$\Rightarrow |A| = 2 \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ 2 & 4 \end{vmatrix} + 0 \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 2(12 - 2) - 1(0 - 4) + 0(0 - 6) = 20 + 4 + 0 = 24 \neq 0$$

Thus, A is non-singular. To find the adjoint of A we find cofactors of all the elements of A .

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = (-1)^2(12 - 2) = (1)(10) = 10$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 2 & 4 \end{vmatrix} = (-1)^3(0 - 4) = (-1)(-4) = 4$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} = (-1)^4(0 - 6) = (1)(-6) = -6$$

Key Facts



- $A^{-1} \neq \frac{1}{A}$
- If the inverse of matrix A exists then it is unique.

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = (-1)^3(4 - 0) = (-1)(4) = -4$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = (-1)^4(8 - 0) = (1)(8) = 8$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = (-1)^5(2 - 2) = (-1)(0) = 0$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = (-1)^4(2 - 0) = (1)(2) = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (-1)^5(4 - 0) = (-1)(4) = -4$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = (-1)^6(6 - 0) = (1)(6) = 6$$

$$\text{Now } \text{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 10 & -4 & 2 \\ 4 & 8 & -4 \\ -6 & 0 & 6 \end{bmatrix}$$

$$\text{And } A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{24} \begin{bmatrix} 10 & -4 & 2 \\ 4 & 8 & -4 \\ -6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{10}{24} & -\frac{4}{24} & \frac{2}{24} \\ \frac{4}{24} & \frac{8}{24} & -\frac{4}{24} \\ -\frac{6}{24} & \frac{0}{24} & \frac{6}{24} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5/12 & -1/6 & 1/12 \\ 1/6 & 1/3 & -1/6 \\ -1/4 & 0 & 1/4 \end{bmatrix}$$

2.3.6 Verification of the Result $(AB)^{-1} = B^{-1}A^{-1}$

If A and B are square matrices of the same order then $(AB)^{-1} = B^{-1}A^{-1}$. To verify this, consider two matrices A and B of the same order.

$$\text{Let } A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

For L.H.S.

$$AB = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3+8 & 6+6 \\ 1+16 & 2+12 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ 17 & 14 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 11 & 12 \\ 17 & 14 \end{vmatrix} = (11)(14) - (17)(12) = 154 - 204 = -50$$

$$\text{And } \text{adj}(AB) = \begin{bmatrix} 14 & -12 \\ -17 & 11 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB) \text{ so}$$

$$(AB)^{-1} = \frac{1}{-50} \begin{bmatrix} 14 & -12 \\ -17 & 11 \end{bmatrix} = \begin{bmatrix} \frac{14}{-50} & \frac{-12}{-50} \\ \frac{-17}{-50} & \frac{11}{-50} \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} -\frac{7}{25} & \frac{6}{25} \\ \frac{17}{50} & -\frac{11}{50} \end{bmatrix} \quad (1)$$

 **Key Facts**
 $(AB)^{-1} = B^{-1}A^{-1}$ is known
as reversal law of inverse.

For R.H.S.

$$|A| = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = (3)(4) - (1)(2) = 12 - 2 = 10$$

$$\text{adj}(A) = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \text{ and}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4/10 & -2/10 \\ -1/10 & 3/10 \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{bmatrix}$$

$$\text{And } |B| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = (1)(3) - (4)(2) = 3 - 8 = -5$$

$$\text{adj}(B) = \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} \text{ and}$$

$$B^{-1} = \frac{1}{|B|} \text{adj}(B) = \frac{1}{-5} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 3/-5 & -2/-5 \\ -4/-5 & 1/-5 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 \\ 4/5 & -1/5 \end{bmatrix}$$

$$\begin{aligned} B^{-1}A^{-1} &= \begin{bmatrix} -3/5 & 2/5 \\ 4/5 & -1/5 \end{bmatrix} \begin{bmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{3}{5}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)\left(-\frac{1}{10}\right) & \left(-\frac{3}{5}\right)\left(-\frac{1}{5}\right) + \left(\frac{2}{5}\right)\left(\frac{3}{10}\right) \\ \left(\frac{4}{5}\right)\left(\frac{2}{5}\right) + \left(-\frac{1}{5}\right)\left(-\frac{1}{10}\right) & \left(\frac{4}{5}\right)\left(-\frac{1}{5}\right) + \left(-\frac{1}{5}\right)\left(\frac{3}{10}\right) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{6}{25} - \frac{2}{25} & \frac{3}{25} + \frac{3}{25} \\ \frac{8}{25} + \frac{1}{50} & -\frac{4}{25} - \frac{3}{50} \end{bmatrix} = \begin{bmatrix} -\frac{8}{25} & \frac{6}{25} \\ \frac{17}{50} & -\frac{11}{50} \end{bmatrix} \quad (2) \end{aligned}$$

From (1) and (2) we have $(AB)^{-1} = B^{-1}A^{-1}$

Exercise 2.3

1. Evaluate the determinant of the following matrices.

(i) $\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{vmatrix}$

(ii) $\begin{vmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} i & 3 & -2i \\ 1 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$

(iv) $\begin{vmatrix} 2+i & 1 & i \\ 0 & 2 & 1 \\ -3i & 1 & 6 \end{vmatrix}$

2. Evaluate the determinants of the following matrices using cofactor method.

(i) $\begin{vmatrix} 3 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 3 & -1 \\ -1 & 0 & 2 \\ 3 & 1 & 4 \end{vmatrix}$

- (iii) $\begin{bmatrix} 2i & 6 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 3i \end{bmatrix}$ (iv) $\begin{bmatrix} 1-i & 2 & 1+i \\ 3 & 1 & 4 \\ 0 & 2 & 3 \end{bmatrix}$
3. Determine which of the following matrices are singular and which are non-singular.
- (i) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ -4 & 1 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 1 \\ -1 & 5 & 1 \end{bmatrix}$
- (iii) $\begin{bmatrix} 3i & 1 & 2 \\ -4 & 1 & i \\ 2 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & -i & 1 \\ i & 3 & -2 \\ -2+i & i+3 & -3 \end{bmatrix}$
4. Find the value of λ , so that the given matrices are singular.
- (i) $\begin{bmatrix} \lambda & 1 & 3 \\ 2 & 1 & 8 \\ 0 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} \lambda & 2 & 0 \\ 2 & 1 & 3 \\ \lambda & 2 & 1 \end{bmatrix}$
- (iii) $\begin{bmatrix} \lambda & i & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2+i & 1 & 6 \\ 2 & \lambda & 1 \\ 3 & 0 & 2 \end{bmatrix}$
5. Find the multiplicative inverse of the following matrices if it exists by adjoint method.
- (i) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -4 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix}$
- (iii) $\begin{bmatrix} i & 0 & 1 \\ 2i & -1 & -i \\ 1 & 0 & 4i \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & -i & i \\ 2 & 1 & -3i \\ 4i & 2 & 6 \end{bmatrix}$
6. If $A = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 0 \\ 2 & 1 & 6 \end{bmatrix}$ then find A^{-1} and hence show that $AA^{-1} = A^{-1}A = I_3$.
7. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ in each of the following.
- (i) $A = \begin{bmatrix} 2 & 1 \\ 8 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$
- (ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$
- (iii) $A = \begin{bmatrix} 2 & -i & 6 \\ 1 & 2 & i \\ -i & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
- (iv) $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

2.4 Properties of Determinants

Here we will discuss some important properties of determinants which will help us to find the value of a determinant. For convenience we will consider the determinant of the square matrix A of order 3×3 i.e. if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

Property 1:

$$|A| = |A^t|$$

Proof:

$$|A^t| = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

Expanding from C_1 ; we have,

$$\begin{aligned} |A^t| &= a_{11} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= |A| \end{aligned}$$

Example:

$$\begin{aligned} \text{If } A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 6 \end{bmatrix} \text{ then } |A| &= \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 6 \end{vmatrix} \\ &= 1 \begin{vmatrix} 3 & 0 \\ 1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & 0 \\ 2 & 6 \end{vmatrix} + 0 \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} \\ &= 1(18 - 0) - 2(0 - 0) + 0 = 18 \end{aligned}$$

And

$$\begin{aligned} |A^t| &= \begin{vmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 0 & 0 & 6 \end{vmatrix} = 1 \begin{vmatrix} 3 & 1 \\ 0 & 6 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} \\ &= 1(18 - 0) - 0 + 2(0 - 0) = 18 \end{aligned}$$

Thus $|A| = |A^t|$

Property 2:

If any two rows (or columns) of a square matrix A are interchanged such that the resulting matrix is B then $|B| = -|A|$.

Proof:

Let we interchange the first and second rows of matrix A ; then the new matrix is:

$$\begin{aligned} B &= \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ \Rightarrow |B| &= \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{21}(a_{12}a_{33} - a_{13}a_{32}) - a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{23}(a_{11}a_{32} - a_{12}a_{31}) \\ &= a_{12}a_{21}a_{33} - a_{13}a_{21}a_{32} - a_{11}a_{22}a_{33} + a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31} \\ &= (-a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32}) + (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (-a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}) \\ &= -a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= -[a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})] \\ \Rightarrow |B| &= -|A| \end{aligned}$$

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ then

$$|A| = 1 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 1(0 - 0) - 2(0 - 0) + 3(4 - 0) = 12$$

By interchanging second and third rows of A ; we have a matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

$$\begin{aligned} \Rightarrow |B| &= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} \\ &= 1(0 - 0) - 2(0 - 0) + 3(0 - 4) = 0 - 0 - 12 = -12 \\ |B| &= -|A| \end{aligned}$$

Property 3:

If any two rows (or columns) of a square matrix are identical then the value of the determinant is zero.

Proof:

Consider a determinant with two identical rows:

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}(a_{22}a_{23} - a_{22}a_{23}) - a_{12}(a_{21}a_{23} - a_{21}a_{23}) + a_{13}(a_{21}a_{22} - a_{21}a_{22}) \\ &= a_{11}(0) - a_{12}(0) + a_{13}(0) = 0 + 0 + 0 = 0 \end{aligned}$$

Example: Consider the determinant $\Delta = \begin{vmatrix} 2 & 1 & 6 \\ 3 & 2 & 0 \\ 3 & 2 & 0 \end{vmatrix}$

Expanding by R_1 :

$$\begin{aligned}\Delta &= 2 \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 3 & 0 \\ 3 & 0 \end{vmatrix} + 6 \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} \\ &= 2(0 - 0) - 1(0 - 0) + 6(6 - 6) \\ &= 2(0) - 1(0) + 6(0) = 0\end{aligned}$$

Property 4:

If we multiply each element of a row or a column with a non-zero scalar k then the resulting matrix is B and $|B| = k|A|$

Proof:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let we multiply each element of row one by a non-zero scalar k then the resulting matrix is

$$B = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow |B| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - ka_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + ka_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= k \left\{ a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \right\}$$

$$= k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Thus $|B| = k|A|$

Example: Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

Let we multiply each element of second row by 3 then the resulting matrix is $B = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

$$\text{Now } |A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} - 3 \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(3 - 0) - 3(6 - 0) + 2(0 - 1) = 3 - 18 - 2 = -17$$

$$\begin{aligned}\text{And } |B| &= \begin{vmatrix} 1 & 3 & 2 \\ 6 & 3 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 1 \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} - 3 \begin{vmatrix} 6 & 0 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 6 & 3 \\ 1 & 0 \end{vmatrix} \\ &= 1(9 - 0) - 3(18 - 0) + 2(0 - 3) = 9 - 54 - 6 = -51 \\ &= 3(-17) = 3|A|\end{aligned}$$

$$\text{i.e., } |B| = 3|A|$$

Property 5:

If matrix A is of the form

$$A = \begin{bmatrix} a_{11} + b_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} & a_{23} \\ a_{31} + b_{31} & a_{32} & a_{33} \end{bmatrix}; \text{ then}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

Proof:

$$|A| = \begin{vmatrix} a_{11} + b_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} & a_{23} \\ a_{31} + b_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding from C_1

$$\begin{aligned}&= (a_{11} + b_{11}) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{21} + b_{21}) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (a_{31} + b_{31}) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + b_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - b_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &\quad + b_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= \left(a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) \\ &\quad + \left(b_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) \\ &|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}\end{aligned}$$

Example: If $A = \begin{bmatrix} 1+2 & 1 & 0 \\ 3-1 & 0 & 2 \\ 2+3 & 2 & 1 \end{bmatrix}$ then

$$\begin{vmatrix} 1+2 & 1 & 0 \\ 3-1 & 0 & 2 \\ 2+3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 2 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\begin{aligned}\text{L.H.S. } \begin{vmatrix} 1+2 & 1 & 0 \\ 3-1 & 0 & 2 \\ 2+3 & 2 & 1 \end{vmatrix} &= \begin{vmatrix} 3 & 1 & 0 \\ 2 & 0 & 2 \\ 5 & 2 & 1 \end{vmatrix} \\ &= 3 \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 0 \\ 5 & 2 \end{vmatrix} = 3(0 - 4) - 1(2 - 10) + 0(4 - 0) \\ &= -12 + 8 + 0 = -4\end{aligned}$$

(1)

$$\begin{aligned}
 \text{R.H.S. } & \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 2 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix} \\
 &= (1 \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix}) + (2 \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + 0 \begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix}) \\
 &= (1(0 - 4) - 1(3 - 4) + 0) + (2(0 - 4) - 1(-1 - 6) + 0) \\
 &= (-4 + 1 + 0) + (-8 + 7 + 0) = -4 \quad (2)
 \end{aligned}$$

Property 6:

If all the elements of a row or a column of a square matrix A are zero then $|A| = 0$

Proof:

Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

So $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expanding from R_1

$$\begin{aligned}
 |A| &= a_{11} \begin{vmatrix} 0 & 0 \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} 0 & 0 \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} 0 & 0 \\ a_{31} & a_{32} \end{vmatrix} \\
 &= a_{11}(0 - 0) - a_{12}(0 - 0) + a_{13}(0 - 0) = 0 + 0 + 0 = 0
 \end{aligned}$$

Example: If $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 9 \end{bmatrix}$ then

$$\begin{aligned}
 |A| &= \begin{vmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 9 \end{vmatrix} \\
 &= 3 \begin{vmatrix} 0 & 0 \\ 3 & 9 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 \\ 1 & 9 \end{vmatrix} + 2 \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} = 3(0 - 0) - 1(0 - 0) + 2(0 - 0) \\
 &= 0 - 0 + 0 = 0
 \end{aligned}$$

Property 7:

If we multiply any row (column) of a square matrix with some scalar k and add the resulting value to the corresponding elements of any other row (column) then the value of the determinant is unchanged.

Proof:

Consider any square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let we multiply R_2 by k and then add the result in R_1 . Resulting matrix is:

$$B = \begin{bmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}
 |B| &= \begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
 &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{21} & ka_{22} & ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
 &= |A| + k \begin{vmatrix} a_{21} & a_{22} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A| + k(0) \\
 \therefore |B| &= |A|
 \end{aligned}$$

$\because R_1$ and R_2 are identical.

Example: Consider the matrix $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \\ 3 & -1 & 2 \end{bmatrix}$ then

$$\begin{aligned}
 |A| &= \begin{vmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \\ 3 & -1 & 2 \end{vmatrix} \\
 &= 3 \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} \\
 &= 3(8 - 1) - 2(2 + 3) + 0(-1 - 12) = 21 - 10 + 0 = 11
 \end{aligned}$$

Let we multiply R_1 By 2 and adding values to the corresponding elements of R_3 ; the resulting matrix is

$$B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \\ 9 & 3 & 2 \end{bmatrix}$$

Now

$$\begin{aligned}
 |B| &= \begin{vmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \\ 9 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ 9 & 2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 4 \\ 9 & 3 \end{vmatrix} \\
 &= 3(8 + 3) - 2(2 + 9) + 0(3 - 36) = 33 - 22 + 0 = 11
 \end{aligned}$$

We conclude that $|A| = |B|$

Property 8:

If a square matrix A is upper triangular or lower triangular or diagonal matrix then $|A|$ is the product of its diagonal elements.

Proof:

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}$$

Expanding by C_1

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - 0 + 0 = a_{11}(a_{22}a_{33} - 0)$$

$$= a_{11}a_{22}a_{33} = \text{product of the diagonal elements}$$

Example: If $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 2 \end{bmatrix}$ then

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} - 0 + 0 = 3(2 - 0) = 6$$

$$= (3)(2)(1) = \text{product of diagonal elements}$$

2.4.2 Evaluation of Determinants Without Expansion

Example: Without expansion show that $\begin{vmatrix} a-2l & b-2m & c-2n \\ l & m & n \\ a & b & c \end{vmatrix} = 0$

Solution: L.H.S

$$\begin{vmatrix} a-2l & b-2m & c-2n \\ l & m & n \\ a & b & c \end{vmatrix} = \begin{vmatrix} a & b & c \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} -2l & -2m & -2n \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ l & m & n \\ a & b & c \end{vmatrix} - 2 \begin{vmatrix} l & m & n \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$= 0 - 2(0) = 0$$

Example: If $a + b + c = 0$, then without expanding show that

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Solution: L.H.S = $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$$= \begin{vmatrix} 2a+2b+2c & 2a+2b+2c & 2a+2b+2c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \text{ by } R_1 + (R_2 + R_3)$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(0) & 2(0) & 2(0) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Example: Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$

Solution:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = \begin{vmatrix} a^2\left(1+\frac{1}{a^2}\right) & ab & ac \\ ab & b^2\left(1+\frac{1}{b^2}\right) & bc \\ ac & bc & c^2\left(1+\frac{1}{c^2}\right) \end{vmatrix}$$

$$= abc \begin{vmatrix} a\left(1+\frac{1}{a^2}\right) & b & c \\ a & b\left(1+\frac{1}{b^2}\right) & c \\ a & b & c\left(1+\frac{1}{c^2}\right) \end{vmatrix}$$

Taking out common
a from R_1 , b from R_2
and c from R_3 .

$$= a^2 b^2 c^2 \begin{vmatrix} \left(1+\frac{1}{a^2}\right) & 1 & 1 \\ 1 & \left(1+\frac{1}{b^2}\right) & 1 \\ 1 & 1 & \left(1+\frac{1}{c^2}\right) \end{vmatrix}$$

Taking out common
a from C_1 , b from C_2
and c from C_3 .

$$= a^2 b^2 c^2 \begin{vmatrix} \frac{1}{a^2} & 0 & 1 \\ 0 & \frac{1}{b^2} & 1 \\ -\frac{1}{c^2} & -\frac{1}{c^2} & 1+\frac{1}{c^2} \end{vmatrix}$$

By $C_1 - C_3$
 $C_2 - C_3$

Expanding from R_1

$$\begin{aligned} &= a^2 b^2 c^2 \left(\frac{1}{a^2} \begin{vmatrix} \frac{1}{b^2} & 1 \\ -\frac{1}{c^2} & 1+\frac{1}{c^2} \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ -\frac{1}{c^2} & 1+\frac{1}{c^2} \end{vmatrix} + 1 \begin{vmatrix} 0 & \frac{1}{b^2} \\ -\frac{1}{c^2} & -\frac{1}{c^2} \end{vmatrix} \right) \\ &= a^2 b^2 c^2 \left[\frac{1}{a^2} \left\{ \frac{1}{b^2} \left(1+\frac{1}{c^2} \right) + \frac{1}{c^2} \right\} - 0 + 1 \left(0 + \frac{1}{b^2 c^2} \right) \right] \\ &= a^2 b^2 c^2 \left(\frac{1}{a^2 b^2} + \frac{1}{a^2 b^2 c^2} + \frac{1}{a^2 c^2} + \frac{1}{b^2 c^2} \right) \\ &= c^2 + 1 + b^2 + a^2 = 1 + a^2 + b^2 + c^2 \end{aligned}$$

Example: Prove that $\begin{vmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{vmatrix} = (1-a^3)^2$

Solution:

$$\begin{vmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{vmatrix} = \begin{vmatrix} 1+a+a^2 & 1+a+a^2 & 1+a+a^2 \\ a & 1 & a^2 \\ a^2 & a & 1 \end{vmatrix}$$

By $R_1 + (R_2 + R_3)$

$$\begin{aligned}
 &= (1+a+a^2) \begin{vmatrix} 1 & 1 & 1 \\ a & 1 & a^2 \\ a^2 & a & 1 \end{vmatrix} \quad \text{Taking out common from } R_1 \\
 &= (1+a+a^2) \begin{vmatrix} 1 & 0 & 0 \\ a & 1-a & a^2-a \\ a^2 & a-a^2 & 1-a^2 \end{vmatrix} \quad \begin{array}{l} \text{By } C_2 - C_1 \\ C_3 - C_1 \end{array} \\
 &\text{Expanding from } R_1 \\
 &= (1+a+a^2) (1 \begin{vmatrix} 1-a & a^2-a \\ a-a^2 & 1-a^2 \end{vmatrix} - 0 + 0) \\
 &= (1+a+a^2) \begin{vmatrix} 1-a & a^2-a \\ a-a^2 & 1-a^2 \end{vmatrix} \\
 &= (1+a+a^2)(1-a)(1-a) \begin{vmatrix} 1 & -a \\ a & 1+a \end{vmatrix} \quad \text{Taking out common from } C_1 \text{ and } C_2 \\
 &= (1+a+a^2)(1-a)(1-a)(1+a+a^2) \\
 &= (1-a^3)(1-a^3) = (1-a^3)^2
 \end{aligned}$$

Exercise 2.4

1. Without expansion show that:

$$\begin{aligned}
 \text{(i)} \quad & \begin{vmatrix} 9 & 27 & 36 \\ 18 & 54 & 24 \\ 27 & 81 & 28 \end{vmatrix} = 0 \quad \text{(ii)} \quad \begin{vmatrix} 1/a & bc & b+c \\ 1/b & ac & a+c \\ 1/c & ab & a+b \end{vmatrix} = 0 \quad \text{(iii)} \quad \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0 \\
 \text{(iv)} \quad & \begin{vmatrix} \sin^2 \alpha & 1 & \cos^2 \alpha \\ \tan^2 \alpha & \sec^2 \alpha & 1 \\ -\operatorname{cosec}^2 \alpha & -\cot^2 \alpha & 1 \end{vmatrix} = 0 \quad \text{(v)} \quad \begin{vmatrix} (a-b)^3 & a^3-b^3 & ab(a-b) \\ (c-d)^3 & c^3-d^3 & cd(c-d) \\ (e-f)^3 & e^3-f^3 & ef(e-f) \end{vmatrix} = 0 \\
 \text{(vi)} \quad & \begin{vmatrix} x & -z & 0 \\ 0 & y & -x \\ -y & 0 & z \end{vmatrix} = 0 \quad \text{(vii)} \quad \begin{vmatrix} (a-b)^2 & (a+b)^2 & ab \\ (c-d)^2 & (c+d)^2 & cd \\ (e-f)^2 & (e+f)^2 & ef \end{vmatrix} = 0
 \end{aligned}$$

2. Using the properties of the determinants prove the following.

$$\begin{aligned}
 \text{(i)} \quad & \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = -2(x^3 + y^3) \\
 \text{(ii)} \quad & \begin{vmatrix} a & b-c & b+c \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a+b+c)(a^2 + b^2 + c^2) \\
 \text{(iii)} \quad & \begin{vmatrix} na_1 + b_1 & na_2 + b_2 & na_3 + b_3 \\ nb_1 + c_1 & nb_2 + c_2 & nb_3 + c_3 \\ nc_1 + a_1 & nc_2 + a_2 & nc_3 + a_3 \end{vmatrix} = (n^3 + 1) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
 \text{(iv)} \quad & \begin{vmatrix} x & x^2 & 1+ax^3 \\ y & y^2 & 1+ay^3 \\ z & z^2 & 1+az^3 \end{vmatrix} = (1+axyz)(x-y)(y-z)(z-x)
 \end{aligned}$$

- (v) $\begin{vmatrix} 2ab & 1+a^2-b^2 & 2b \\ 2a & -2b & 1-a^2-b^2 \\ 1-a^2+b^2 & 2ab & -2a \end{vmatrix} = (1+a^2+b^2)^3$
- (vi) $\begin{vmatrix} 3a & 1 & 2a+1 \\ 2a+1 & 1 & a+2 \\ 3 & 1 & 2 \end{vmatrix} = (a-1)(a-2)$
- (vii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$
- (viii) $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^2$
- (ix) $\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$
- (x) $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$
- (xi) If $|AB| = |A| \cdot |B|$ and $|A^{-1}| = 1/|A|$ then for a square matrix of order 3×3 prove that $|adj A| = |A|^2$
- (xii) If A is of order 3×3 such that $|adj A| = 64$ then find $|A^{-1}|$.
- (xiii) If a, b, c are real numbers and $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$. Show that either $a+b+c = 0$ or $a = b = c$.

2.5. Rows and Columns Operations

2.5.1 Rows and Columns Operations on Matrices

Elementary Row Operations

The following elementary row operations can be performed on a matrix.

- We can interchange any two rows of the matrix. If we interchange the i th row with the j th row of the matrix then it is denoted by R_{ij} .
- We can multiply any row by a non-zero scalar k with the i th row then it is denoted by kR_i .
- We can add a multiple of any row to the corresponding values of any other row. If we add k -times of the j th row to the i th row then it is denoted by $R_i + kR_j$.

Elementary Column Operations

The following elementary column operations can be performed on a matrix.

- We can interchange any two columns of the matrix. If we interchange the i th column with the j th column of the matrix then it is denoted by C_{ij} .

- (ii) We can multiply any column by a non-zero scalar k with the i th column then it is denoted by kC_i .
- (iii) We can add a multiple of any other column to the corresponding values of any other column. If we add k -times of the j th column to the i th column then it is denoted by $C_i + kC_j$.

2.5.2 Echelon Form of a Matrix

Any matrix which has the following properties is known as in Echelon form (row Echelon form).

- (i) If a row does not consist entirely of zeros, then the first non-zero number in the row is 1; we call this leading 1.
- (ii) If there are many rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
- (iii) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
- (iv) Each column that contains a leading 1 has zeros below 1.

2.5.3 Reduced Echelon Form of a Matrix

Any matrix which has the following properties is known as in Reduced Echelon form.

- (i) If a row does not consist entirely of zeros, then the first non-zero number in the row is 1; we call this leading 1.
- (ii) If there are many rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
- (iii) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
- (iv) Each column that contains a leading 1 has zero everywhere else in that column.

Example: Reduce the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ into the echelon form.

Solution: $A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$$\sim R \quad \begin{bmatrix} 1 & 0 & 2 \\ -2 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

by R_{13}

$$\sim R \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 5 \\ 0 & 1 & -4 \end{bmatrix}$$

by $R_2 + 2R_1$
 $R_3 - 3R_1$

$$\sim R \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 4 & 5 \end{bmatrix}$$

by R_{23}

Key Facts

A matrix in reduced echelon form is also in echelon form; but a matrix in echelon form may not be in reduced echelon form.



$$\sim R \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 21 \end{bmatrix} \quad \text{by } R_3 - 4R_2$$

$$\sim R \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{by } \frac{1}{21}R_3$$

Which is the required echelon form of matrix A .

Example: Write the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ into the reduced echelon form.

Solution: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

$$\sim R \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -3 & -6 & -9 \end{bmatrix} \quad \begin{array}{l} \text{by } R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\sim R \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{by } -R_2 \text{ and } -\frac{1}{3}R_3$$

$$\sim R \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{by } R_1 - 2R_2 \\ R_3 - R_2 \end{array}$$

Which is the reduced echelon form.

2.5.4 Rank of a Matrix

Using Row Operations to Find the Rank of a Matrix

To find the rank of a matrix, find its echelon (or reduced echelon) form. The number of non-zero rows in its echelon form is called the rank (or row rank) of the matrix.

Example: Find the rank of the matrix $\begin{bmatrix} 2 & 5 & 7 \\ 1 & 2 & -1 \\ -3 & -6 & 3 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 2 & -1 \\ -3 & -6 & 3 \end{bmatrix}$

$$\sim R \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 7 \\ -3 & -6 & 3 \end{bmatrix} \quad \text{by } R_{12}$$

$$\sim R \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

by $R_2 - 2R_1$
 $R_3 + 3R_1$

Which is the echelon form of the matrix. The number of non-zero rows is 2.

Thus $\text{Rank}(A) = 2$

2.5.5 Using Row Operation to Find the Inverse of a Non-Singular Matrix

Row operations can be performed on a non-singular matrix A to find its inverse. For this consider an identity matrix I of same order as that of A . Write A and I parallel to each other. Now perform some row operations on A and I so that matrix A reduce to I , consequently the matrix I will also reduce to some new matrix which is the inverse of A .

We can also perform column operations to find A^{-1} .

Example: Find A^{-1} ; if $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ by using row operations.

Solution: $|A| = \begin{vmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}$

$$= 2 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 4 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 2(6 - 0) - 1(8 - 1) + 0(0 - 3)$$

$$= 12 - 7 + 0 = 5 \neq 0$$

So A is non-singular. Now consider

$$[A|I] = \begin{bmatrix} 2 & 1 & 0 & : & 1 & 0 & 0 \\ 4 & 3 & 1 & : & 0 & 1 & 0 \\ 1 & 0 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim R \begin{bmatrix} 1 & 0 & 2 & : & 0 & 0 & 1 \\ 4 & 3 & 1 & : & 0 & 1 & 0 \\ 2 & 1 & 0 & : & 1 & 0 & 0 \end{bmatrix}$$

by R_{13}

$$\sim R \begin{bmatrix} 1 & 0 & 2 & : & 0 & 0 & 1 \\ 0 & 3 & -7 & : & 0 & 1 & -4 \\ 0 & 1 & -4 & : & 1 & 0 & -2 \end{bmatrix}$$

by $R_2 - 4R_1$
 $R_3 - 2R_1$

$$\sim R \begin{bmatrix} 1 & 0 & 2 & : & 0 & 0 & 1 \\ 0 & 1 & -4 & : & 1 & 0 & -2 \\ 0 & 3 & -7 & : & 0 & 1 & -4 \end{bmatrix}$$

by R_{23}

$$\sim R \begin{bmatrix} 1 & 0 & 2 & : & 0 & 0 & 1 \\ 0 & 1 & -4 & : & 1 & 0 & -2 \\ 0 & 0 & 5 & : & -3 & 1 & 2 \end{bmatrix}$$

by $R_3 - 3R_2$

$$\sim R \begin{bmatrix} 1 & 0 & 2 & : & 0 & 0 & 1 \\ 0 & 1 & -4 & : & 1 & 0 & -2 \\ 0 & 0 & 1 & : & -3/5 & 1/5 & 2/5 \end{bmatrix}$$

by $\frac{1}{5}R_3$

Thus

$$A^{-1} = \begin{bmatrix} 6/5 & -2/5 & 1/5 \\ -7/5 & 4/5 & -2/5 \\ -3/5 & 1/5 & 2/5 \end{bmatrix}$$

Exercise 2.5

1. First reduce each of the following matrices into echelon form then into reduced echelon form.

(i) $\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 9 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & 8 \\ -3 & 1 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & -4 & 3 \\ 4 & 1 & 8 \\ 7 & 3 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 9 & 8 \end{bmatrix}$

(vi) $\begin{bmatrix} 0 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

2. Find the rank of each of the following matrices.

(i) $\begin{bmatrix} 5 & 9 & 3 \\ 3 & -5 & 6 \\ 2 & 10 & 6 \end{bmatrix}$

(ii) $\begin{bmatrix} -1 & -2 & 3 \\ -1 & 2 & -1 \\ -5 & 2 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 6 \\ 4 & -1 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 3 \\ 2 & 9 \\ 1 & 6 \end{bmatrix}$

3. With the help of row operations, find the inverse of the following matrices if it exists.

Also verify your answer by showing that $AA^{-1} = A^{-1}A = I$.

(i) $\begin{bmatrix} 0 & -1 & -1 \\ -1 & 3 & 0 \\ 1 & -1 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 5 \\ -3 & 0 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} -5 & 2 & 3 \\ -1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 & 3 \\ 3 & 2 & 4 \\ 6 & -1 & 2 \end{bmatrix}$

2.6 Solving System of Linear Equations

Linear Equation

An equation of the form $a_1x_1 + a_2x_2 = k_1$; where a_1, a_2 and k_1 are constants and at least one of a_1 and a_2 is non-zero is called a linear equation in two variables x_1 and x_2 .

Similarly, the equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = k_2$; where a_1, a_2, a_3 and k_2 are constants and at least one of a_1, a_2 and a_3 is non-zero is called a linear equation in three variables x_1, x_2 and x_3 . In the same manner we can extend this for n -number of variables.

System of Linear Equations

When we deal with more than one linear equation at the same time; then it is called system of linear equations. We divide the system of linear equations into two categories:

- (i) Homogeneous system of linear equations.
- (ii) Non-homogeneous system of linear equations.

2.6.1 Homogeneous and Non-homogeneous Linear Equations

Homogeneous System of Linear Equations

Consider the following system of linear equations

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3$$

If $k_1 = k_2 = k_3 = 0$; then the system is called homogeneous system of linear equations.

Non- Homogeneous System of Linear Equations

For the following system of linear equations

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3$$

If at least one of k_1, k_2 and k_3 is non-zero then the system is called non-homogeneous system of equations.

2.6.2 Solution of System of Linear Equations

The values of the variables involved in the system of linear equations which when substituted in any equation of the system the equation is satisfied; is known as the solution of the system.

A system may have no solution or unique solution or infinite number of solutions.

Solution of Homogeneous System of Linear Equations

Consider a system of homogeneous equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

This system may be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

So, we write $AX = O$

Observe that each equation of the system is satisfied if we take $x = 0$; $y = 0$; $z = 0$. So, $(0, 0, 0)$ is the solution of the homogeneous system of linear equations. Since this solution always exists for all systems of the homogeneous equations thus it is called trivial solutions of the system. All solutions other than trivial solution are known as non-trivial solutions of the system.

Observe that, if the coefficients matrix A is non-singular then A^{-1} exists; so

$$(1) \Rightarrow A^{-1}(AX) = A^{-1}(O)$$

$$\Rightarrow (A^{-1}A)X = O$$

$$\Rightarrow IX = O$$

$$\Rightarrow X = O$$

Key Facts

Condition for the system of homogeneous linear equations to have non-trivial solution is that $|A| = 0$.

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x = 0, y = 0, z = 0$$

i.e., The system has a trivial solution.

The system of homogeneous linear equations may have non-trivial solution if $|A| = 0$.

Example: How many solutions does the following system of homogeneous linear equations has?

$$3x - 2y + z = 0 \quad (1)$$

$$2x + y - 3z = 0 \quad (2)$$

$$x - y + z = 0 \quad (3)$$

Solution:

The coefficients matrix is:

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} |A| &= 3 \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 3(1 - 3) + 2(2 + 3) + 1(-2 - 1) \\ &= -6 + 10 - 3 = 1 \neq 0 \end{aligned}$$

This system has only trivial solution.

Example: Solve the homogeneous system of linear equations:

$$x + 3y + 2z = 0 \quad (1)$$

$$2x - y + 3z = 0 \quad (2)$$

$$x - 4y + z = 0 \quad (3)$$

Solution:

The coefficients matrix is:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 1 & -4 & 1 \end{bmatrix}$$

$$|A| = 1 \begin{vmatrix} -1 & 3 \\ -4 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix}$$

$$|A| = 1(-1 + 12) - 3(2 - 3) + 2(-8 + 1)$$

$$|A| = 11 + 3 - 14 = 0$$

So the system has non-trivial solution.

Multiplying equation (1) by 2 then subtracting equation (2) from it, we get:

$$(1) \Rightarrow 2x + 6y + 4z = 0$$

$$(2) \Rightarrow 2x - y + 3z = 0$$

$$\begin{array}{r} - \quad + \quad - \\ \hline 7y + z = 0 \end{array}$$

(4)

Subtracting equation (3) from equation (1), we have:

$$(1) \Rightarrow x + 3y + 2z = 0$$

$$(3) \Rightarrow x - 4y + z = 0$$

$$\begin{array}{r} x + 3y + 2z = 0 \\ -(x - 4y + z = 0) \\ \hline 7y + z = 0 \end{array}$$

(5)

Now equations (4) and (5) are identical

Put $z = t$ in equation (4).

$$\Rightarrow 7y + t = 0 \Rightarrow 7y = -t \Rightarrow y = -\frac{1}{7}t$$

Substituting these values in equation (1), we have:

$$x + 3\left(-\frac{1}{7}t\right) + 2t = 0$$

$$\Rightarrow x - \frac{3}{7}t + 2t = 0 \Rightarrow x + \frac{11}{7}t = 0$$

$$\Rightarrow x = -\frac{11}{7}t$$

Thus $\left(-\frac{11}{7}t, -\frac{1}{7}t, t\right)$ are the infinite many solutions. By assigning different values to t we will have different solutions.

Consistent System of Equations

A system of linear equations which has at least one solution is called consistent system of equations.

In-consistent System of Equations

A system of linear equations which has no solution at all is called in-consistent system of equations.

2.6.3 Solution of Non-Homogeneous System of Linear Equations

Consider a non-homogeneous system of linear equations:

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3$$

where, at least one of k_1, k_2 and k_3 is non-zero.

The above system in matrix form may be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$
 $\Rightarrow AX = B$



Remember
System of homogeneous linear equations is always consistent since it has at least trivial solution.

Consistency Criteria

A system of homogeneous linear equations is consistent if $\text{Rank } A = \text{Rank } A_b$. The system is inconsistent if $\text{Rank } A \neq \text{Rank } A_b$.

If $\text{Rank } A = \text{Rank } A_b = \text{number of unknowns}$, then the system has a unique solution.

If $\text{Rank } A = \text{Rank } A_b < \text{number of unknowns}$, then system has infinite many solutions.

Augmented Matrix

For a given system of linear equations, a matrix consisting of the coefficients of the unknowns together with the constants on the right side of equations is called an augmented matrix. It is usually denoted by A_b . For the above system of linear equations the augmented matrix is:

$$A_b = \begin{bmatrix} a_1 & b_1 & c_1 & : & k_1 \\ a_2 & b_2 & c_2 & : & k_2 \\ a_3 & b_3 & c_3 & : & k_3 \end{bmatrix}$$

Methods to Solve a Non-Homogeneous System of Equations

To solve a system of 3 - by - 3 non-homogeneous system of linear equations, we use the following methods.

- Matrix inversion method
- Gauss elimination method (echelon form)
- Gauss Jordan method (reduced echelon form)
- Cramer's rule

Matrix Inversion Method

Consider the non-homogeneous system of linear equations:

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3$$

In matrix form this system may be written as:

$$AX = B$$

If A is invertible (i.e.; non-singular) then A^{-1} exists; so

$$A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

Example: Solve the system of non-homogeneous linear equation by matrix inversion method.

$$2x + 3y - z = 1; \quad x - y + z = 3; \quad x + 2y - z = 1$$

Solution:

For this system of equations; we have

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\text{and } |A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & - & 1 \\ 1 & 2 & -1 \end{vmatrix} = 2 \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ = 2(1 - 2) - 3(-1 - 1) + 1(2 + 1) = -2 + 6 - 3 = 1 \neq 0$$

This system is consistent. Now to find A^{-1} , we calculate the cofactors of each element.

$$A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = (-1)^2(1 - 2) = -1$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1)^3(-1 - 1) = 2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = (-1)^4(2 + 1) = 3$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} = (-1)^3(-3 + 2) = 1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = (-1)^4(-2 + 1) = -1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^5(4 - 3) = -1$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = (-1)^4(3 - 1) = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = (-1)^5(2 + 1) = -3$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = (-1)^6(-2 - 3) = -5$$

$$\text{adj}A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -3 \\ 3 & -1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{1} \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -3 \\ 3 & -1 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -3 \\ 3 & -1 & -5 \end{bmatrix}$$

Since $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -3 \\ 3 & -1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 3 + 2 \\ 2 - 3 - 3 \\ 3 - 3 - 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -5 \end{bmatrix}$$

$\therefore x = 4; y = -4$ and $z = -5$ is its solution.

Gauss Elimination Method (Echelon Form)

In this method, we reduce the associated augmented matrix for a given system of linear equations to its echelon form.

Example: Solve the system of equations by using Gauss elimination method

$$2x_1 - 3x_2 + 4x_3 = 1; x_1 + 2x_2 - x_3 = 2; 3x_1 + 5x_2 - 3x_3 = 5$$

Solution:

The associated augmented matrix is:

$$A_b = \left[\begin{array}{ccc|c} 2 & -3 & 4 & 1 \\ 1 & 2 & -1 & 2 \\ 3 & 5 & -3 & 5 \end{array} \right]$$

First, we reduce it into echelon form.

$$\sim R \begin{bmatrix} 1 & 2 & -1 & : & 2 \\ 2 & -3 & 4 & : & 1 \\ 3 & 5 & -3 & : & 5 \end{bmatrix}$$

by R_{12}

$$\sim R \begin{bmatrix} 1 & 2 & -1 & : & 2 \\ 0 & -7 & 6 & : & -3 \\ 0 & -1 & 0 & : & -1 \end{bmatrix}$$

by $R_2 - 2R_1$
 $R_3 - 3R_1$

$$\sim R \begin{bmatrix} 1 & 2 & -1 & : & 2 \\ 0 & -1 & 0 & : & -1 \\ 0 & -7 & 6 & : & -3 \end{bmatrix}$$

by R_{23}

$$\sim R \begin{bmatrix} 1 & 2 & -1 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & -7 & 6 & : & -3 \end{bmatrix}$$

by $-1R_2$

$$\sim R \begin{bmatrix} 1 & 2 & -1 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 6 & : & 4 \end{bmatrix}$$

by $R_3 + 7R_2$

$$\sim R \begin{bmatrix} 1 & 2 & -1 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & \frac{2}{3} \end{bmatrix}$$

by $\frac{1}{6}R_3$

Which is the echelon form of A_b . From the last row we have:

$$0x_1 + 0x_2 + x_3 = \frac{2}{3}$$

$$\Rightarrow x_3 = \frac{2}{3}$$

Key Facts



If $\text{Rank } A = \text{Rank } A_b = 3$
Then the system has a unique solution.

From second row we have:

$$0x_1 + 1x_2 + 0x_3 = 1$$

$$\Rightarrow x_2 = 1$$

From the first row, we have:

$$x_1 + 2x_2 - x_3 = 2$$

$$x_1 + 2(1) - \frac{2}{3} = 2 \Rightarrow x_1 + \frac{4}{3} = 2$$

$$\Rightarrow x_1 = \frac{2}{3}$$

$\therefore x_1 = \frac{2}{3}; \quad x_2 = 1; \quad x_3 = \frac{2}{3}$ is the solution of the system.

Gauss Jordan Method (Reduced Echelon Form)

In this method, we reduce the associated augmented matrix into reduced echelon form for the given system of non-homogeneous linear equations.

Example: Solve the system of given non-homogeneous linear equations

$$2x - 3y + 5z = 2; \quad x + 4y - 2z = 1; \quad 4x + 5y + z = 4,$$

by using Gauss-Jordan method.

Solution:

The associated augmented matrix is:

$$A_b = \begin{bmatrix} 2 & -3 & 5 & : & 2 \\ 1 & 4 & -2 & : & 1 \\ 4 & 5 & 1 & : & 4 \end{bmatrix}$$

First, we reduced it into the reduced echelon form.

$$\sim R \begin{bmatrix} 1 & 4 & -2 & : & 1 \\ 2 & -3 & 5 & : & 2 \\ 4 & 5 & 1 & : & 4 \end{bmatrix} \quad \text{by } R_{12}$$

$$\sim R \begin{bmatrix} 1 & 4 & -2 & : & 1 \\ 0 & -11 & 9 & : & 0 \\ 0 & -11 & 9 & : & 0 \end{bmatrix} \quad \begin{array}{l} \text{by } R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\sim R \begin{bmatrix} 1 & 4 & -2 & : & 1 \\ 0 & 1 & -\frac{9}{11} & : & 0 \\ 0 & -11 & 9 & : & 0 \end{bmatrix} \quad \text{by } \frac{-1}{11} R_{12}$$

$$\sim R \begin{bmatrix} 1 & 0 & 14/11 & : & 1 \\ 0 & 1 & -9/11 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad \begin{array}{l} \text{by } R_1 - 4R_2 \\ R_3 + 11R_2 \end{array}$$

Observe that $\text{Rank } A = \text{Rank } A_b = 2$, which is less than the number of unknowns. Therefore system has infinite many solutions. From the last row, we have:

$$0x + 0y + 0z = 0$$

This equation is true for all values of the unknowns involved; so let $z = t$.

From second row, we have:

$$\begin{aligned} 0x + y - \frac{9}{11}z &= 0 \\ \Rightarrow y - \frac{9}{11}t &= 0 \Rightarrow y = \frac{9}{11}t \end{aligned}$$

From row one, we have:

$$\begin{aligned} x + 0y + \frac{14}{11}z &= 1 \\ \Rightarrow x + \frac{14}{11}t &= 1 \Rightarrow x = 1 - \frac{14}{11}t \end{aligned}$$

Thus, $x = 1 - \frac{14}{11}t$; $y = \frac{9}{11}t$; $z = t$ provide us infinite many solutions by assigning different values to the parameter 't'.

Cramer's Rule

Consider a system of non-homogeneous linear equations:

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3$$

The system may be written in matrix form as $AX = B$.

If A is non-singular then $|A| \neq 0$ and A^{-1} exists.

$$\therefore A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

Since,

$$A^{-1} = \frac{1}{|A|} (\text{adj} A)$$

So

$$X = \frac{1}{|A|} (\text{adj} A) B$$

$$\Rightarrow X = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} k_1 A_{11} + k_2 A_{21} + k_3 A_{31} \\ k_1 A_{12} + k_2 A_{22} + k_3 A_{32} \\ k_1 A_{13} + k_2 A_{23} + k_3 A_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{k_1 A_{11} + k_2 A_{21} + k_3 A_{31}}{|A|} \\ \frac{k_1 A_{12} + k_2 A_{22} + k_3 A_{32}}{|A|} \\ \frac{k_1 A_{13} + k_2 A_{23} + k_3 A_{33}}{|A|} \end{bmatrix}$$

Comparing the elements, we have

$$x = \frac{k_1 A_{11} + k_2 A_{21} + k_3 A_{31}}{|A|}$$

$$y = \frac{k_1 A_{12} + k_2 A_{22} + k_3 A_{32}}{|A|}$$

$$z = \frac{k_1 A_{13} + k_2 A_{23} + k_3 A_{33}}{|A|}$$

Now

$$\begin{aligned} & k_1 A_{11} + k_2 A_{21} + k_3 A_{31} \\ &= k_1 \left((-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \right) + k_2 \left((-1)^{2+1} \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \right) + k_3 \left((-1)^{3+1} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \right) \\ &= k_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - k_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + k_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \end{aligned}$$

Thus,

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{|A|}$$

Similarly,

$$y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{|A|}$$

Remember



Like matrix inversion method; Cramer rule can be used only if A is non-singular.

$$z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{|A|}$$

Example: Solve the given system of non-homogeneous linear equations

$$2x - 3y + 5z = 1; \quad x + y + 2z = 3; \quad 3x - 2y - 4z = 0$$

by Cramer's rule.

Solution:

The above system may be written as $AX = B$; where,

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 1 & 2 \\ 3 & -2 & -4 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 1 & 2 \\ 3 & -2 & -4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$$

$$= 2(-4 + 4) + 3(-4 - 6) + 5(-2 - 3) = 0 - 30 - 25$$

$$= -55 \neq 0$$

So, A is non-singular.

$$\therefore x = \frac{\begin{vmatrix} 1 & -3 & 5 \\ 3 & 1 & 2 \\ 0 & -2 & -4 \end{vmatrix}}{|A|} = \frac{1 \begin{vmatrix} 1 & 2 \\ 0 & -4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} + 5 \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix}}{-55}$$

$$= \frac{1(-4 + 4) + 3(-12 - 0) + 5(-6 - 0)}{-55}$$

$$= \frac{-66}{-55} = \frac{6}{5}$$

$$y = \frac{\begin{vmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \\ 3 & 0 & -4 \end{vmatrix}}{|A|} = \frac{2 \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix}}{-55}$$

$$= \frac{2(-12 - 0) - 1(-4 - 6) + 5(0 - 9)}{-55}$$

$$= \frac{-24 + 10 - 45}{-55} = \frac{59}{55}$$

$$z = \frac{\begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & 3 \\ 3 & -2 & 0 \end{vmatrix}}{|A|} = \frac{-2 \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}}{-55} = \frac{2(0+6) + 3(0-9) + 1(-2-3)}{-55}$$

$$= \frac{12 - 27 - 5}{-55} = \frac{4}{11}$$

Application of Matrices

Matrices are used in many disciplines. For example, in cryptography. We explain the process of encryption and decryption by means of an example.

Suppose that the sender and receiver consider messages in alphabets *A* to *Z* only, both assign the numbers 1 to 26 to the letters *A* to *Z* respectively, and the number 0 to a blank space. For simplicity, the sender employs a key as post-multiplication by a non-singular matrix of order 3 of his own choice. The receiver uses post-multiplication by the inverse of the matrix which has been chosen by the sender.

Let the encoding matrix be

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Let the message to be sent by the sender be "WELCOME".

Since the key is taken as the operation of post-multiplication by a square matrix of order 3, the message is cut into pieces (WEL), (COM), (E), each of length 3, and converted into a sequence of row matrices of numbers:

$$[23 \ 5 \ 12], [3 \ 15 \ 13], [5 \ 0 \ 0].$$

Note that, we have included two zeros in the last row matrix. The reason is to get a row matrix with 5 as the first entry.

Next, we encode the message by post-multiplying each row matrix as given below:

Uncoded Row matrix	Encoding Matrix	Coded row Matrix
$[23 \ 5 \ 12]$	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$[45 \ -28 \ 23]$

$[3 \ 15 \ 13]$	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$[46 \ -18 \ 3]$
$[5 \ 0 \ 0]$	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$[5 \ -5 \ 5]$

So the encoded message is $[45 \ -28 \ -23][46 \ -18 \ 3][5 \ -5 \ 5]$

The receiver will decode the message by the reverse key, post-multiplying by the inverse of A.

So the decoding matrix is

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

The receiver decodes the coded message as follows:

Coded Row matrix	Decoded Matrix	Decoded Row matrix
$[45 \ -28 \ -23]$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$	$[23 \ 5 \ 12]$
$[46 \ -18 \ 3]$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$	$[3 \ 15 \ 13]$
$[5 \ -5 \ 5]$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$	$[5 \ 0 \ 0]$

So, the sequence of decoded row matrices is $[23 \ 5 \ 12]$, $[3 \ 15 \ 13]$, $[5 \ 0 \ 0]$.

Thus, the receiver reads the message as "WELCOME".

Exercise 2.6

1. Solve the following system of homogeneous linear equations for non-trivial solution if exists.

(i) $2x_1 - 3x_2 + 4x_3 = 0$

$$x_1 - 2x_2 + 3x_3 = 0$$

$$4x_1 + x_2 - 6x_3 = 0$$

(iii) $x_1 + x_2 - 3x_3 = 0$

$$3x_1 - 2x_2 + x_3 = 0$$

$$4x_1 - x_2 - 2x_3 = 0$$

(ii) $2x_1 - 3x_2 + 4x_3 = 0$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 - 4x_2 + 3x_3 = 0$$

(iv) $5x_1 + 6x_2 - 7x_3 = 0$

$$2x_1 - x_2 + x_3 = 0$$

$$x_1 + 2x_2 + 2x_3 = 0$$

2. Find the value of λ for which the following system of homogeneous linear equations may have non-trivial solution. Also solve the system for value of λ .

(i) $2x_1 - \lambda x_2 + x_3 = 0$

$$2x_1 + 3x_2 - x_3 = 0$$

$$3x_1 - 2x_2 + 4x_3 = 0$$

(ii) $x_1 - 4x_2 + 3x_3 = 0$

$$2x_1 + \lambda x_2 + x_3 = 0$$

$$x_1 - 2x_2 + \lambda x_3 = 0$$

3. Solve the following system of linear equations by Gauss elimination method.

(i) $2x + 3y + 4z = 2$

$$2x + y + z = 5$$

$$3x - 2y + z = -3$$

(iii) $2x + z = 2$

$$2y - z = 3$$

$$x + 3y = 5$$

(ii) $5x - 2y + z = 2$

$$2x + 2y + 6z = 1$$

$$3x - 4y - 5z = 3$$

(iv) $x + 2y + 5z = 4$

$$3x - 2y + 2z = 3$$

$$5x - 8y - 4z = 1$$

4. Solve the following system of linear equations by Gauss-Jordan method.

(i) $2x_1 - x_2 - x_3 = 2$

$$3x_1 - 4x_2 + 3x_3 = 7$$

$$4x_1 + 2x_2 - 5x_3 = 10$$

(ii) $2x_1 - 3x_2 + 7x_3 = 1$

$$4x_1 + 5x_2 - 3x_3 = 4$$

$$10x_1 - 4x_2 + 18x_3 = 7$$

(iii) $x_1 + x_2 + x_3 = 3$

$$2x_1 - 3x_2 + 2x_3 = 7$$

$$4x_1 + 2x_2 - 5x_3 = 10$$

(iv) $2x_1 - 7x_2 + 10x_3 = 1$

$$x_1 + 2x_2 - 4x_3 = 8$$

$$2x_1 - 11x_2 + 13x_3 = 7$$

5. Solve the following system of linear equations by using Cramer's rule.

(i) $x_1 + x_2 + 2x_3 = 8$

$$-x_1 - 2x_2 + 3x_3 = 1$$

$$3x_1 - 7x_2 + 4x_3 = 10$$

(ii) $2x_1 + 2x_2 + x_3 = 0$

$$-2x_1 + 5x_2 + 2x_3 = 1$$

$$8x_1 + x_2 + 4x_3 = -1$$

(iii) $-2x_2 + 3x_3 = 1$

$$3x_1 + 6x_2 - 3x_3 = -2$$

$$6x_1 + 6x_2 + 3x_3 = 5$$

(iv) $2x_1 + x_2 + 3x_3 = 1$

$$x_1 - 2x_2 + x_3 = 2$$

$$3x_1 - 4x_2 - x_3 = 4$$

6. Solve the following system of linear equations by matrix inversion method.

(i) $5x + 3y + z = 6$

$2x + y + 3z = 19$

$x + 2y + 4z = 25$

(ii) $x + 2y - 3z = 5$

$2x - 3y + 2z = 1$

$-x + 2y - 5z = -3$

(iii) $-x + 3y - 5z = 0$

$2x + 4y - 6z = 1$

$x - 2y + 3z = 3$

(iv) $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$

$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$

$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

7. If $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$; find A^{-1} and hence solve the system of equations.

$3x + 4y + 7z = 14; 2x - y + 3z = 4; x + 2y - 3z = 0.$

8. Determine the value of λ for which the following system has no solution, unique solution or infinitely many solutions.

$x + 2y - 3z = 4; 3x - y + 5z = 2; 4x + y + (\lambda^2 - 14)z = \lambda + 2$

9. Show that the system of equations

$2x - y + 3z = \alpha; 3x + y - 5z = \beta; -5x - 5y + 21z = \gamma$

is inconsistent if $\gamma \neq 2\alpha - 3\beta$.

10. By making use of matrix of order 2 by 2 and 3 by 3 encode and decode the following words:

a. PAKISTAN

b. ISLAMABAD

c. COLLEGE

I have Learnt

- Applying matrix operations (addition/subtraction and multiplication of (matrices) with real and complex entries.
- Evaluating determinants of 3×3 matrix by using cofactors and properties of determinants.
- Using row operations to find the inverse and the rank of a matrix.
- Explaining a consistent and inconsistent system of linear equations and demonstrate through examples.
- Solving a system of 3 by 3 nonhomogeneous linear equations by using matrix inversion method and Cramer's Rule.
- Solving a system of three homogeneous linear equations in three unknowns using the Gaussian elimination method.
- Applying concepts of matrices to real world problems such as (graphic design, data encryption, seismic analysis, cryptography, transformation of geometric shapes, social network analysis).

Review Exercise

1. Select the best matching option.

(i) If order of A is $m \times n$ and order of B is $n \times p$ then order of AB is:

- (a) $n \times p$ (b) $m \times p$ (c) $p \times m$ (d) $n \times n$

(ii) If A is a row matrix of order $1 \times n$ then order of $A^t A$ is:

- (a) $1 \times n$ (b) $n \times 1$ (c) 1×1 (d) $n \times n$

(iii) For an element a_{ij} of a square matrix A :

- (a) $a_{ij} = (-1)^{i+j} A_{ij}$ (b) $a_{ij} = (-1)^{i+j} M_{ij}$ (c) $\frac{A_{ij}}{M_{ij}} = (-1)^{i+j}$ (d) $a_{ij} = M_{ij}$

(iv) If A is any matrix then A and A^t are always conformable for:

- (a) Addition (b) multiplication (c) subtraction (d) all of these

(v) If A is a square matrix of order 3×3 and $|A| = 3$ then value of $|adj A|$ is:

- (a) 3 (b) $1/3$ (c) 9 (d) 6

(vi) For the square matrix A of order 3×3 with $|A| = 9$; $A_{21} = 2$; $A_{22} = 3$; $A_{23} = -1$; $a_{21} = 1$; $a_{23} = 2$, the value of a_{22} is:

- (a) 2 (b) 3 (c) 9 (d) -1

(vii) System of homogeneous linear equations has non-trivial solution if:

- (a) $|A| > 0$ (b) $|A| < 0$ (c) $|A| = 0$ (d) $|A| \neq 0$

(viii) For non-homogeneous system of equations; the system is inconsistent if:

- (a) $\text{Rank } A = \text{Rank } A_b$ (b) $\text{Rank } A \neq \text{Rank } A_b$
(c) $\text{Rank } A < \text{no. of variables}$ (d) $\text{Rank } A_b > \text{no. of variables}$

(ix) For a system of non-homogeneous equations with three variables system will have unique solution if:

- (a) $\text{Rank } A < 3$ (b) $\text{Rank } A_b < 3$
(c) $\text{Rank } A = \text{Rank } A_b = 3$ (d) $\text{Rank } A = \text{Rank } A_b < 3$

(x) A system of non-homogeneous equation having infinite many solutions can be solved by using:

- (a) Inversion method (b) Cramer's rule
(c) Gauss-Jordan method (d) all of these

2. For the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 4 & 9 \\ 2 & 1 & 6 \end{bmatrix}$; find A_{13} , A_{23} and A_{33} ; hence find $|A|$.

3. Prove that if $A^{-1} = A^t$ then $|AA^t| = 1$.

4. Without expanding show that $\begin{vmatrix} a+1 & l & l \\ l & a+1 & l \\ l & l & a+1 \end{vmatrix} = (a+1+2l)(a+1-l)^2$.

5. Find the value of λ so that the following system has infinite many solutions.

$$2x - 3y + z = 1; x - 2y + \lambda z = 2; 3y + z = -1$$