Unit-11

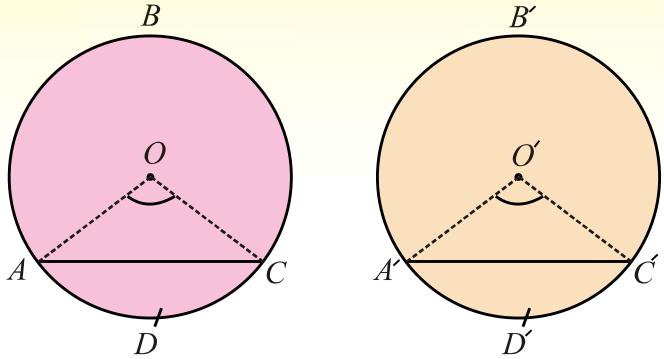
CHORDS AND ARCS

In this unit, students will learn

- ***If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.***
- ***If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.***
- ***Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).***
- ***If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.***

THEOREM 1

11.1(*i*) If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. So that $m\widehat{ADC} = m\widehat{A'D'C'}$

To prove: $m\overline{AC} = m\overline{A'C'}$

Construction: Join O with A, O with C, O' with A' and O' with C'. So that we can form Δ^s OAC and $O'A'C'$.

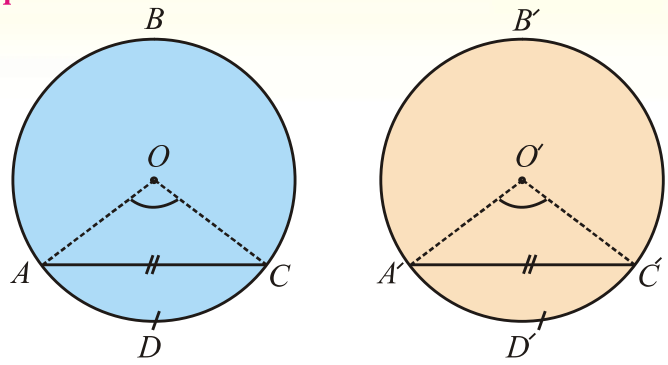
Proof:

Statements	Reasons
In two equal circles $ABCD$ and $A'B'C'D'$ with centres O and O' respectively.	Given
$m\widehat{ADC} = m\widehat{A'D'C'}$	Given
$\therefore$ $m\angle AOC = m\angle A'O'C'$	Central angles subtended by equal arcs of the equal circles.
Now in $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\angle AOC = m\angle A'O'C'$	Already Proved
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$\therefore$ $\Delta AOC \cong \Delta A'O'C'$	S.A.S ≅ S.A.S
and in particular $m\overline{AC} = m\overline{A'C'}$	
Similarly we can prove the theorem in the same circle.	

THEOREM 2

Converse of Theorem 1

11.1(*ii*) If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent. In equal circles or in the same circle, if two chords are equal, they cut off equal arcs.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively.

So that chord $m\overline{AC} = m\overline{A'C'}$.

To prove: $m\widehat{ADC} = m\widehat{A'D'C'}$

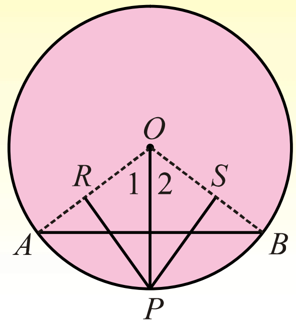
Construction: Join O with A, O with C, O' with A' and O' with C'.

Proof:

Statements	Reasons
In $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$m\,\overline{AC} = m\,\overline{A'C'}$	Given
$\therefore$ $\Delta AOC \cong \Delta A'O'C'$	S.S.S $\cong$ S.S.S
$\Rightarrow$ $m\angle AOC = m\angle A'O'C'$	
Hence $m\widehat{ADC} = m\widehat{A'D'C'}$	Arcs corresponding to equal central angles.

Example 1: A point P on the circumference is equidistant from the radii $\overline{OA}$ and OB.

Prove that $m\,\widehat{AP} = m\,\widehat{BP}$



Given: AB is the chord of a circle with centre O. Point P on the circumference of the circle is equidistant from the radii $\overline{OA}$ and $\overline{OB}$ so that $m\overline{PR} = m\overline{PS}$.

To prove: $m\,\widehat{AP} = m\,\widehat{BP}$

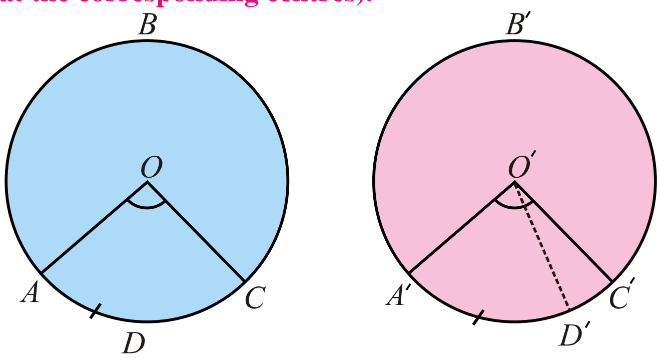
Construction: Join O with P. Write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements	Reasons
In $\angle$rt ΔOPR and $\angle rt\ \Delta OPS$	
$m\,\overline{OP} = m\,\overline{OP}$	Common
$m\,\overline{PR} = m\,\overline{PS}$	Point P is equidistance from radii (Given)
$\therefore$ $\Delta OPR \cong \Delta\ OPS$	(In $\angle rt\Delta^s$ H.S $\cong$ H.S)
So $m\,\angle 1 = m\,\angle 2$	Central angles of a circle
$\Rightarrow$ Chord $AP \cong$ Chord BP	
Hence $m\,\widehat{AP} = m\,\widehat{BP}$	Arcs corresponding to equal chords in a circle.

THEOREM 3

11.1(*iii*) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).



Given: ABC and $A'B'C'$ are two congruent circles with centres O and O' respectively.

So that $\overline{AC} = \overline{A'C'}$

To prove: $\angle AOC \cong \angle A'O'C'$

Construction: Let if possible $m\angle AOC \neq m\angle A'O'C'$ then consider $\angle AOC \cong \angle A'O'D'$

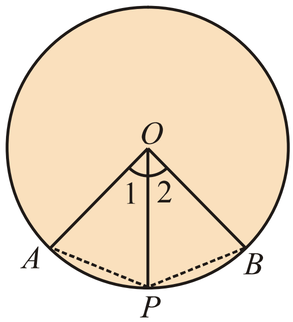
Proof:

Statements			Reasons
	$\angle AOC \cong \angle A'O'D'$		Construction
$\therefore$	$\overset{\frown}{AC} \cong \overset{\frown}{A'D'}$	(i)	Arcs subtended by equal Central angles in congruent circles
	$m\overline{AC} = m\overline{A'D'}$	(ii)	Using Theorem 1
But	$m\overline{AC} = m\overline{A'C'}$	(iii)	Given
$\therefore$	$m\overline{A'C'} = m\overline{A'D'}$		Using (ii) and (iii)
Which is only possible, if C' concides with D'.			
Hence	$m\angle A'O'C' = m\angle A'O'D'$	(iv)	
But	$m\angle AOC = m\angle A'O'D'$	(v)	Construction
$\Rightarrow$	$m\angle AOC = m\angle A'O'C'$		Using (iv) and (v)

Corollary 1. In congruent circles or in the same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2. In congruent circles or in the same circle, unequal arcs will subtend unequal central angles.

Example 1: The internal bisector of a central angle in a circle bisects an arc on which it stands.



Solution: In a circle with centre O. $\overline{OP}$ is an internal bisector of central angle AOB.

To prove: $\overset{\frown}{AP} \cong \overset{\frown}{BP}$

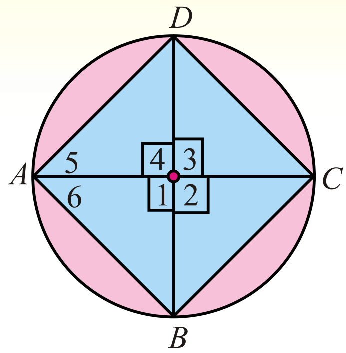
Construction: Draw $\overline{AP}$ and $\overline{BP}$, then write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements		Reasons
	In $\Delta OAP \leftrightarrow \Delta OBP$	
	$m\,\overline{OA} = m\,\overline{OB}$	Radii of the same circle
	$m\angle 1 = m\angle 2$	Given $\overline{OP}$ as an angle bisector of $\angle AOB$
and	$m\,\overline{OP} = m\,\overline{OP}$	Common
		(S.A.S $\cong$ S.A.S)

$\Delta OAP \cong \Delta OBP$	
Hence $\overline{AP} \cong \overline{BP}$	
$\Rightarrow \quad \widehat{AP} \cong \widehat{BP}$	Arcs corresponding to equal chords in a circle.

Example 2: In a circle if any pair of diameters are ⊥ to each other then the lines joining its ends in order, form a square.



Given: $\overline{AC}$ and $\overline{BD}$ are two perpendicular diameters of a circle with centre O. So $ABCD$ is a quadrilateral.

To prove: $ABCD$ is a square

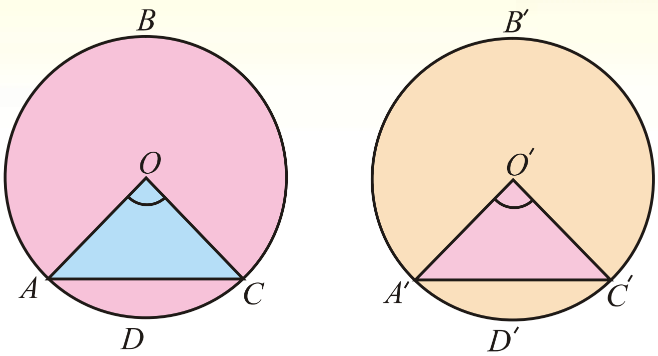
Construction: Write ∠1, ∠2, ∠3, ∠4, ∠5 and ∠6 as shown in the figure.

Proof:

Statements	Reasons
$\overline{AC}$ and $\overline{BD}$ are two ⊥ diameters of a circle with centre O	Given
$\therefore \quad m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4 = 90^o$	Pair of diameters are ⊥ to each other.
$\therefore \quad m\widehat{AB} = m\widehat{BC} = m\widehat{CD} = m\widehat{DA}$	Arcs opposite to the equal central angles in a circle.
$\Rightarrow \quad m\,\overline{AB} = m\,\overline{BC} = m\overline{CD} = m\overline{DA}$ (i)	Chords corresponding to equal arcs.
Moreover $m\angle A = m\angle 5 + m\angle 6$	
$= 45^o + 45^o = 90^o$ (ii)	
Similarly $m\angle B = m\angle C = m\angle D = 90^o$ (iii)	
Hence $ABCD$ is a square	Using (i), (ii) and (iii).

THEOREM 4

11.1(*iv*) If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. $\overline{AC}$ and $\overline{A'C'}$ are chords of circles $ABCD$ and $A'B'C'D'$ respectively and $m\angle AOC = m\angle A'O'C'$

To prove: $m\overline{AC} = m\overline{A'C'}$

Proof:

Statements	Reasons
In $\Delta OAC \longleftrightarrow \Delta O'A'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of congruent circles
$m\angle AOC = m\angle A'O'C'$	Given
$m\overline{OC} = m\overline{O'C'}$	Radii of congruent circles
$\therefore$ $\Delta OAC \cong \Delta O'A'C'$	$SAS \cong SAS$
Hence $m\overline{AC} = m\overline{A'C'}$	

EXERCISE 11.1

1. In a circle two equal diameters $\overline{AB}$ and $\overline{CD}$ intersect each other.
   Prove that $m\,\overline{AD} = m\,\overline{BC}$.
2. In a circle prove that the arcs between two parallel and equal chords are equal.
3. Give a geometric proof that a pair of bisecting chords are the diameters of a circle.
4. If C is the mid point of an arc ACB in a circle with centre O. Show that line segment OC bisects the chord AB.

MISCELLANEOUS EXERCISE 11

1. Multiple Choice Questions

Four possible answers are given for the following questions.
Tick (✓) the correct answer.

(*i*) A 4 cm long chord subtends a central angle of 60^o. The radial segment of this circle is:

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

(*ii*) The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be:

(*a*) 30^o (*b*) 45^o (*c*) 60^o (*d*) 75^o

(*iii*) Out of two congruent arcs of a circle, if one arc makes a central angle of 30^o then the other arc will subtend the central angle of:

(*a*) 15^o (*b*) 30^o (*c*) 45^o (*d*) 60^o

(*iv*) An arc subtends a central angle of 40^o then the corresponding chord will subtend a central angle of:

(*a*) 20^o (*b*) 40^o (*c*) 60^o (*d*) 80^o

(*v*) A pair of chords of a circle subtending two congruent central angles is:

(*a*) congruent (*b*) incongruent (*c*) over lapping (*d*) parallel

(*vi*) If an arc of a circle subtends a central angle of 60^o, then the corresponding chord of the arc will make the central angle of:

(*a*) 20^o (*b*) 40^o (*c*) 60^o (*d*) 80^o

(*vii*) The semi circumference and the diameter of a circle both subtend a central angle of:

(*a*) 90^o (*b*) 180^o (*c*) 270^o (*d*) 360^o

(*viii*) The chord length of a circle subtending a central angle of 180^o is always:

(*a*) less than radial segment (*b*) equal to the radial segment
(*c*) double of the radial segment (*d*) none of these

(*ix*) If a chord of a circle subtends a central angle of 60^o, then the length of the chord and the radial segment are:

(*a*) congruent (*b*) incongruent (*c*) parallel (*d*) perpendicular

(*x*) The arcs opposite to incongruent central angles of a circle arc always:

(*a*) congruent (*b*) incongruent (*c*) parallel (*d*) perpendicular

SUMMARY

- The boundary traced by a moving point in a circle is called its circumference whereas any portion of the circumference will be known as an arc of the circle.
- The straight line joining any two points of the circumference is called a chord of the circle.
- The portion of a circle bounded by an arc and a chord is known as the segment of a circle.
- The circular region bounded by an arc of a circle and its two corresponding radial segments is called a sector of the circle.
- A straight line, drawn from the centre of a circle bisecting a chord is perpendicular to the chord and conversely perpendicular drawn from the centre of a circle on a chord, bisects it.
- If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.