VECTORS AND EQUILIBRIUM

PHYSICAL QUANTITIES

Those quantities which are measured or detected by their physical effect are called as physical quantities.

TYPES OF PHYSICAL QUANTITIES

There are two types of physical quantities given as

(a) Scalar Quantities
(b) Vector Quantities

SCALAR-QUANTITIES

Those physical quantities which are completely described by their magnitudes with suitable units and have no particular directions, called scalar quantities.

For Example; Time, Speed, Length, Mass, Work, Energy, Power, Electric flux, etc.

VECTOR QUANTITIES

Those physical quantities which are completely described by their magnitudes with suitable units and have particular directions, called as vector quantities.

For Example; Displacement, Velocity, Acceleration, Linear momentum, Force, Weight, Torque, etc.

MAGNITUDE

The numerical value of a physical quantity is called its magnitude.

For example; if a force is 100 N, then 100 is the magnitude of force.

UNIT

Unit has magnitude unity i-e; one (1).

DEFINITION: The fundamental name of a physical quantity by which it is known in the whole world, called unit.

For example, if a force is 100 N, then 100 is magnitude of force and newton (N) is its unit.

Reresentation Of A Vector

A vector is represented by two ways.

(a) Symbolical Representation.

(b) Graphical Representation.

Symbolical Representation

Symbolically, a vector is usually represented by bold face letter or capital letter with an arrow mark over its head or a line drawn below the letter.

For example, a vector A can be represented as

$$\mathbf{A} \text{ or } \vec{A} \text{ or } \underline{A}$$

The magnitude of the vector can be represented by taking the modulus (| |) of the letter or write the letter without arrow mark or line.

For example, the magnitude of $\vec{A}$ is

$$|\vec{A}| \text{ or } |\underline{A}| \text{ or } A$$

Graphical Representation

Graphically, a vector is represented by an arrow or with a straight line with arrow head at one end. The length of the arrow represents the magnitude of the vector by taking a suitable scale and arrow head represents the direction of the vector according to the directional frame.

The one end of the vector having arrow head is referred as **HEAD** while other end is referred as **TAIL**.

For example, a car is moving with a velocity of 250 Km/h towards North. So

Velocity $= \vec{V} = 250$ Km/h (north)

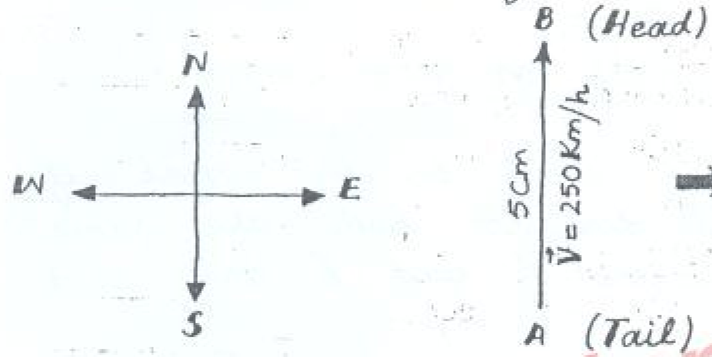
Now we select a suitable scale as

Scale

50 Km/h = 1 Cm

250 Km/h = 5 Cm

And then draw the vector graphically according to the directional frame



$\Rightarrow \vec{V} = \vec{AB} = 250$ Km/h

$= 5$ Cm

Consider another example of a body apply a force of 150 N at an angle of 30° with x-axis. So

Force $= \vec{F} = 150$ N at 30° with X-axis.

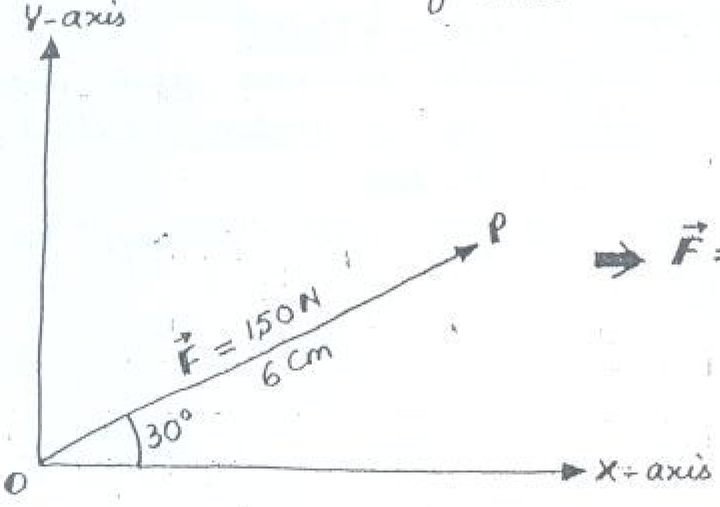
Select a suitable scale as

Scale

25 N = 1 Cm

150 N = 6 Cm

Now draw the vector graphically according to the directional frame



$\Rightarrow \vec{F} = \vec{OP} = 150$ N

$= 6$ Cm

Null or Zero Vector

A vector whose magnitude is zero and arbitrary direction, called null or zero vector.

For example, the subtraction and vector product of two equal vectors is zero. i-e;

$$\vec{A} - \vec{A} = \vec{O}$$

$$\vec{A} \times \vec{A} = \vec{O}$$

Where $\vec{O}$ is null or zero vector.

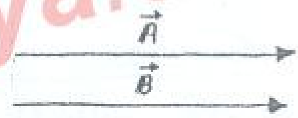
Equal Vectors

The vectors are said to be equal if their magnitude and direction both are same.

For example, vectors $\vec{A}$ and $\vec{B}$ are said to be equal i-e;

$$\vec{A} = \vec{B}$$

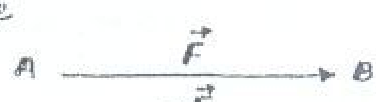
If $A = B$ and $\theta = 0°$



Negative Of A Vector

A vector whose magnitude is equal to the magnitude of a given vector but opposite in direction, called negative of a given vector.

For example, the negative of a vector $\vec{F} = \vec{AB}$ is $-\vec{F} = \vec{BA}$ as shown in figure.



Unit Vector or Directional Vector

A vector whose magnitude is one and used to represent the direction of a vector, called unit vector or directional vector.

It is denoted by a letter with "a cap" or "a hat" (^) over the letter.

Consider a vector $\vec{A}$ given as

$$\vec{A} = |\vec{A}|\,\hat{A}$$

or $\quad \vec{A} = A\hat{A} \longrightarrow (1)$

Where A is the magnitude of $\vec{A}$ and $\hat{A}$ is the unit vector in the direction of $\vec{A}$.

From equation ①, we have

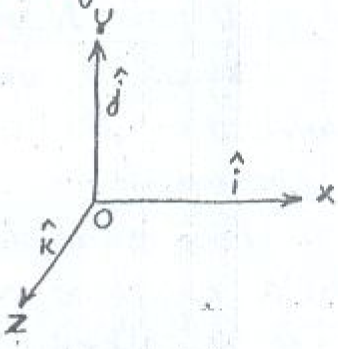
$$\vec{A} = A\hat{A}$$

or
$$\boxed{\hat{A} = \frac{\vec{A}}{A}} \longrightarrow ②$$

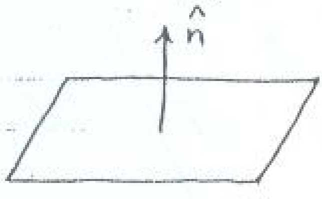
or
$$\boxed{\text{UNIT VECTOR} = \frac{\text{GIVEN VECTOR}}{\text{MAGNITUDE OF GIVEN VECTOR}}}$$

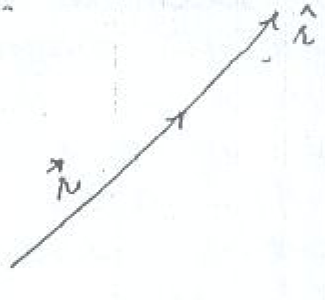
Hence, unit vector can be obtained by dividing the vector with its magnitude.

In case of three dimensional space, the directions along X, Y and Z-axes are represented by unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.



Two unit vectors $\hat{r}$ and $\hat{n}$ are more frequently used. Unit vector $\hat{r}$ represents the direction of position vector $\vec{r}$ while the unit vector $\hat{n}$ represents the direction of a normal drawn on a surface as shown in figure





MULTIPLICATION OF A VECTOR WITH A SCALAR

When a vector is multiplied with a scalar or a number, then its magnitude changes and the direction of the vector may or may not change.

Now we consider two cases:

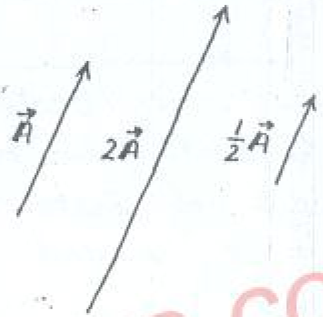
Case (i): When Number Is Positive

When a vector is multiplied with a positive number, then its magnitude changes and direction remains same.

For example, when a vector $\vec{A}$ is multiplied with a positive number n $(n>0)$, then a new vector $n\vec{A}$ is obtained having the same direction as that of $\vec{A}$ and its magnitude becomes n-times the magnitude of $\vec{A}$ i-e;

$$|n\vec{A}| = n|\vec{A}|$$

This effect is shown in the figure.



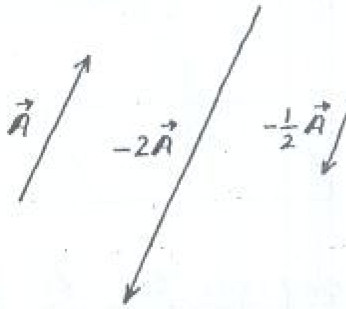
Case (ii): When Number Is Negative

When a vector is multiplied with a negative number, then its magnitude changes and its direction is reversed.

For example, when a vector $\vec{A}$ is multiplied with a negative number n $(n<0)$, then a new vector $n\vec{A}$ is obtained in the opposite direction as that of $\vec{A}$ and its magnitude becomes n-times the magnitude of $\vec{A}$ i-e;

$$|n\vec{A}| = n|\vec{A}|$$

This effect is shown in the figure.



It should be noted that the dimensions of resulting vector will be the same as the product of the dimensions of the two quantities multiplied.

For example, when a vector quantity like velocity is multiplied by a scalar mass, the product is a new vector quantity called momentum $(m\vec{v})$ having the same dimensions as those of mass and velocity.

Resultant Vector

A vector whose effect is equal to the sum of two or more vectors, called resultant vector.

It is denoted by $\vec{R}$.

Consider, two vectors $\vec{A}_1$ and $\vec{A}_2$, then the resultant vector $\vec{R}$ of their possible sum is

$$\vec{R} = \vec{A}_1 + \vec{A}_2$$

Similarly, for n-vectors $\vec{A}_1, \vec{A}_2$, ------, $\vec{A}_n$, then their resultant vector is

$$\vec{R} = \vec{A}_1 + \vec{A}_2 + \text{------} + \vec{A}_n$$

$$\vec{R} = \sum_{i=1}^{i=n} \vec{A}_i$$

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where $\sum$ (sigma) stands for summation or sum up.

ADDITION OF VECTORS. By Graphical Method.

A process in which two or more vectors are added, called addition of vectors.

For parallel or anti-parallel vectors, the vectors are added by simple arithmetic means. For non-parallel vectors, the vectors are not added or subtracted by simple arithematic means. For this process, vectors are added or subtracted by head-to-tail rule.

Head-To-Tail Rule.

First of all, we draw vectors graphically by taking a suitable scale. The first vector draw as it and then second vector is added in the first vector by joining its tail with the head of first vector. Similarly, tail of third vector is joined with the head of second vector and so on. To get resultant vector, draw a vector itself whose tail joined with the tail of first vector and head with the head of last vector. This method is known as Head-To-Tail rule for addition of vectors.

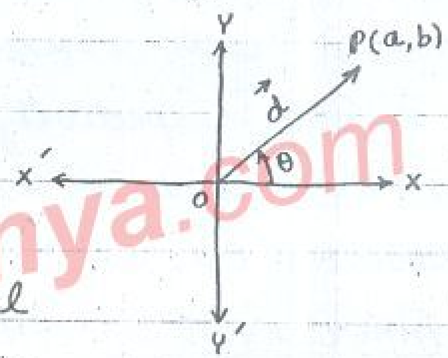
EXAMPLE

Consider two vectors $\vec{A_1}$ and $\vec{A_2}$ are to be added. First we draw them graphically, then head of $\vec{A_1}$ is joined with the tail of $\vec{A_2}$ and for resultant vector $\vec{R}$, joint the head of $\vec{A_2}$ with the tail of $\vec{A_1}$. So that the resultant vector $\vec{R}$ given as

$$\vec{R} = \vec{A_1} + \vec{A_2}$$

RECTANGULAR COORDINATE SYSTEM OR CARTESIAN COORDINATE SYSTEM.

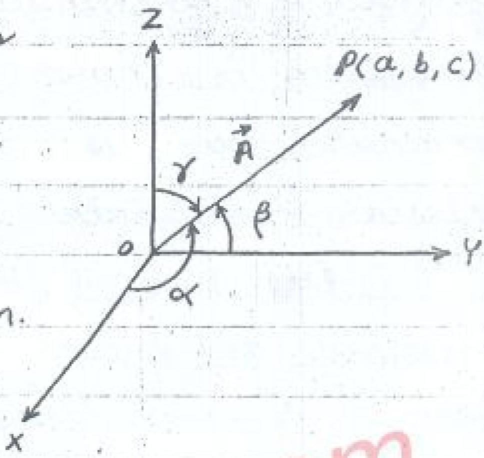
Two lines drawn at right angle or perpendicular to each other are known as frame of refrence or coordinate axes and their point of intersection is known as origin. This system of coordinate axes is called as rectangular or cartesian coordinate system.



One line of the axes is named as X-axis while other as Y-axis. Usually, X-axis is taken as horizental axis with positive direction to the right and negative direction to the left. And Y-axis is taken as vertical axis with the positive direction upward and negative direction downward.

The direction of a vector $\vec{d}$ represented by line $\overrightarrow{OP}$ in XY-plane is denoted by an angle θ with positive X-axis in the anti-clock wise direction as shown in figure. Let point P has coordinates (a, b), it means that if we start from origin, then we reach point P by moving 'a' units along positive X-axis and 'b' units along positive Y-axis.

The direction of a vector in space requir three axis which are mutually perpendicular lines, known as x-axis, Y-axis and Z-axis as shown. Now the direction of a vector $\vec{A}$ in space is denoted by three angles, say α, β and γ makes with x-axis, Y-axis and Z-axis respectively as shown. Here the point P has three coordinates (a, b, c).

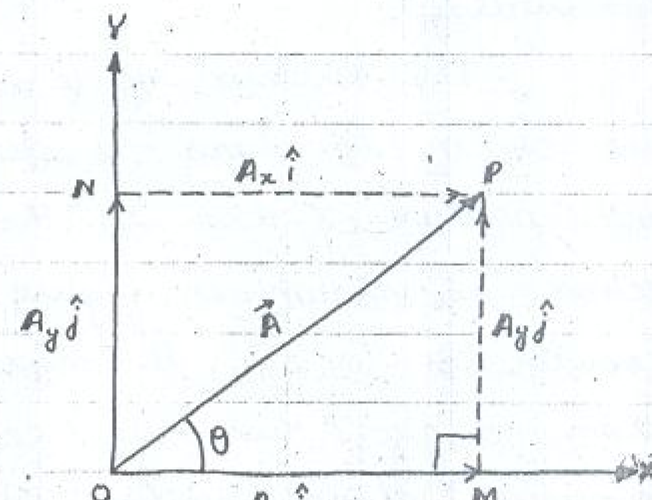


RECTANGULAR COMPONENTS OF A VECTOR

The components of a vector which are at right angle or perpendicular to each other are called rectangular components of a vector.

EXPLANATION

Consider a vector $\vec{A}$ represented by line $\vec{OP}$ making angle θ with x-axis as shown in figure. For rectangular components of $\vec{A}$, draw perpendiculars from $\vec{A}$ on X-axis and Y-axis at points M and N respectively, such that



$$\vec{OM} = \vec{A_x} = \vec{NP}$$
$$\vec{ON} = \vec{A_y} = \vec{MP}$$

Where $\vec{A}_x$ is the component of $\vec{A}$ acting along X-axis, called X-component or Horizontal component. $\vec{A}_x$ is the effective part of $\vec{A}$ acting along X-axis and can be obtained by drawing projection $\vec{OM}$ from $\vec{A}$ on X-axis.

Similarly, $\vec{A}_y$ is the component of $\vec{A}$ acting along Y-axis, called Y-component or vertical component. $\vec{A}_y$ is the effective part of $\vec{A}$ acting along Y-axis and can be obtained by drawing projection $\vec{ON}$ from $\vec{A}$ on Y-axis.

Since $\vec{A}_x$ and $\vec{A}_y$ are at right angle to each other, so these are called rectangular components of $\vec{A}$. Also $\vec{A}_x$ and $\vec{A}_y$ are the sides of rectangle OMPN, so these are called as rectangular components.

Now from right angle ΔOMP, we have

$$\vec{OP} = \vec{OM} + \vec{MP}$$

or $$\vec{A} = \vec{A}_x + \vec{A}_y$$

or $$\vec{A} = A_x\hat{i} + A_y\hat{j} \longrightarrow (1)$$

DETERMENITION OF RECTANGULAR COMPONENTS

From right angle ΔOMP, we have

$$\frac{OM}{OP} = \frac{Base}{Hyp} = \cos\theta$$

or $$\frac{A_x}{A} = \cos\theta$$

or $$\boxed{A_x = A\cos\theta} \longrightarrow (2)$$

In vector form

$$\vec{A}_x = A\cos\theta \cdot \hat{i}$$

Also

$$\frac{MP}{OP} = \frac{Perp}{Hyp} = \sin\theta$$

or $$\frac{A_y}{A} = \sin\theta$$

or $$\boxed{A_y = A\sin\theta} \longrightarrow (3)$$

In vector form

$$\vec{A}_y = A\sin\theta \cdot \hat{j}$$

Determinition Of A Vector From Rectangular Components

Magnitude Of Vector $\vec{A}$

From right angle ΔOMP, using Pythagorean theorem

$$(Hyp)^2 = (Base)^2 + (Perp)^2$$

or $$(OP)^2 = (OM)^2 + (MP)^2$$

or $$A^2 = A_x^2 + A_y^2$$

Taking square root

$$\sqrt{A^2} = \sqrt{A_x^2 + A_y^2}$$

$$\boxed{A = \sqrt{A_x^2 + A_y^2}}$$

Which is the magnitude of vector $\vec{A}$.

Direction Of Vector $\vec{A}$

From right angle ΔOMP, we have

$$\tan\theta = \frac{Perp}{Base} = \frac{MP}{OP}$$

or $$\tan\theta = \frac{A_y}{A_x}$$

or $$\boxed{\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)}$$

Which is direction of vector $\vec{A}$

ADDITION OF VECTORS BY RECTANGULAR COMPONENTS

Consider two vectors $\vec{A}$ and B represented by lines $\vec{OM}$ and $\vec{ON}$ making angles θ_1 and θ_2 respectively with x-axis as shown in figure. These vectors are added by head-to-tail to get the resultant vector $\vec{R}$ represented by line $\vec{OP}$ making certain angle θ with x-axis as shown.

From figure, the resultant vector $\vec{R}$ is given as

$$\vec{OP} = \vec{OM} + \vec{MP}$$

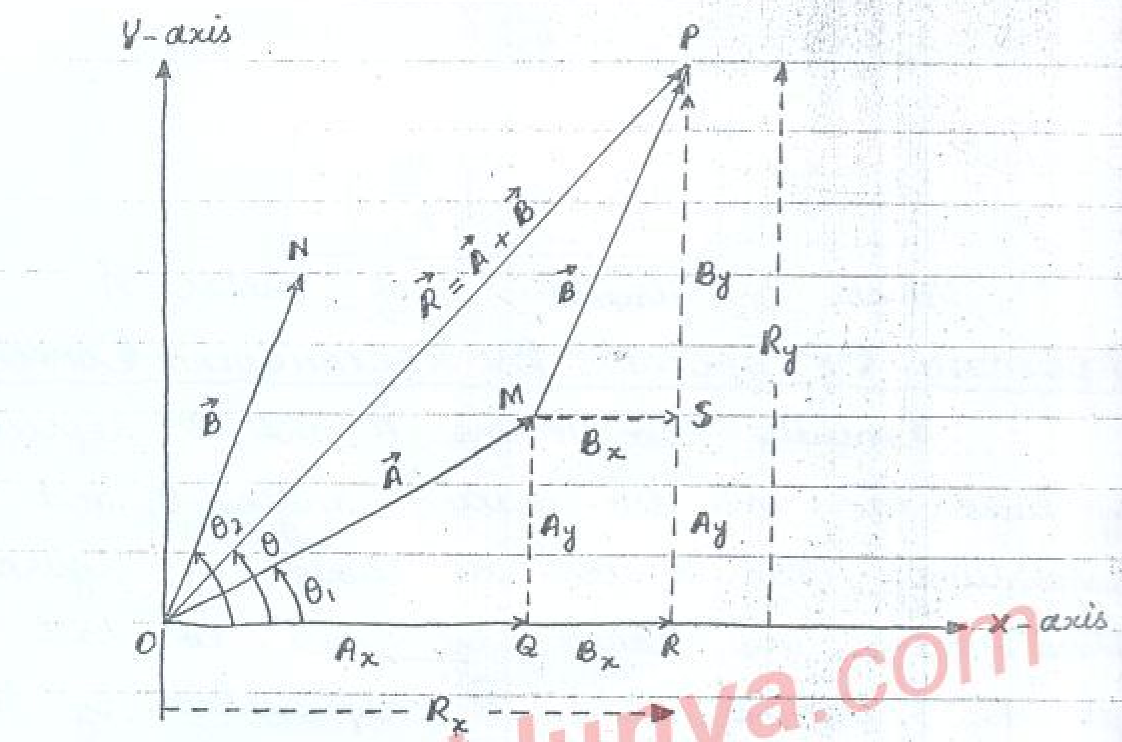
or $$\vec{R} = \vec{A} + \vec{B} \longrightarrow (1)$$

For rectangular components of the vectors $\vec{A}$ and $\vec{B}$, draw perpendiculars from $\vec{A}$ and $\vec{B}$ on x-axis at points Q and R respectively. Also draw a perpendicular from point M on line RP at point S. Hence, vectors in their rectangular components are

$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{i} + A_y\hat{j}$$

$$\vec{B} = \vec{B}_x + \vec{B}_y = B_x\hat{i} + B_y\hat{j}$$

and $$\vec{R} = \vec{R}_x + \vec{R}_y = R_x\hat{i} + R_y\hat{j}$$



Such that

$A_x = OQ$, $A_y = QM = RS$

$B_x = MS = QR$, $B_y = SP$

$R_x = OR$, $R_y = RP$

It is clear from figure that the x-component of resultant vector is

$$OR = OQ + QR$$

or $$OR = OQ + MS \qquad (\because QR = MS)$$

or $$R_x = A_x + B_x \longrightarrow ②$$

This equation shows that the sum of magnitudes of x-components of two vectors is equal to the magnitude of x-component of resultant vector.

Similarly, the Y-component of resultant vector is given as

$$RP = RS + SP$$

or $$RP = QM + SP \qquad (\because RS = QM)$$

or $$R_y = A_y + B_y \longrightarrow (3)$$

This equation shows that the sum of magnitudes of Y-components of two vectors is equal to the magnitude of Y-component of resultant vector.

MAGNITUDE OF $\vec{R}$

The magnitude of resultant vector $\vec{R}$ can be obtained by using Pythagorean's theorem on right angle ΔORP, we have

$$(Hyp)^2 = (Base)^2 + (Perp)^2$$

or $$(OP)^2 = (OR)^2 + (RP)^2$$

or $$R^2 = R_x^2 + R_y^2$$

Taking square root on both sides

$$R = \sqrt{R_x^2 + R_y^2}$$

Put values of R_x and R_y from eq. (2) and (3)

$$\boxed{R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}}$$

DIRECTION OF $\vec{R}$

From right angle ΔORP, we find the direction of resultant vector as

$$\tan\theta = \frac{Perp}{Base} = \frac{RP}{OR}$$

or $$\tan\theta = \frac{R_y}{R_x}$$

or $$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

Again put values of R_x and R_y

$$\boxed{\theta = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right)}$$

Addition Of Any Number Of Vectors

To find the resultant vector of any number of vectors $\vec{A}, \vec{B}, \vec{C}$, -------, we proceed as

The X-component of resultant vector is

$$R_x = A_x + B_x + C_x + \text{--------}$$

And Y-component of resultant vector is

$$R_y = A_y + B_y + C_y + \text{--------}$$

Now the magnitude of $\vec{R}$ is given as

$$R = \sqrt{R_x^2 + R_y^2}$$

$$R = \sqrt{(A_x + B_x + C_x + \text{------})^2 + (A_y + B_y + C_y + \text{------})^2}$$

And the direction of $\vec{R}$ is given as

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

$$\theta = \tan^{-1}\left(\frac{A_y + B_y + C_y + \text{------}}{A_x + B_x + C_x + \text{------}}\right)$$

RULES FOR FINDING RESULTANT VECTOR BY RECTANGULAR COMPONENTS

The addition of vectors to get the resultant vector by rectangular components consists of following steps.

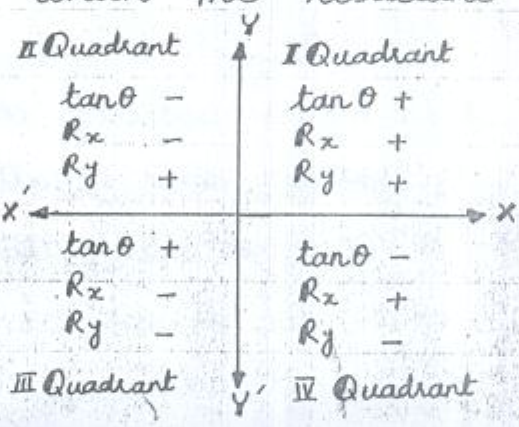
1. Find X and Y - components of all given vectors.
2. Find the X - component of resultant vector R_x by adding X - components of all the vectors.
3. Find the Y - component of resultant vector R_y by adding Y - components of all the vectors.
4. Find the magnitude of resultant vector $\vec{R}$ by using the relation

$$R = \sqrt{R_x^2 + R_y^2}$$

5. Find the direction of resultant vector $\vec{R}$ by using the relation

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

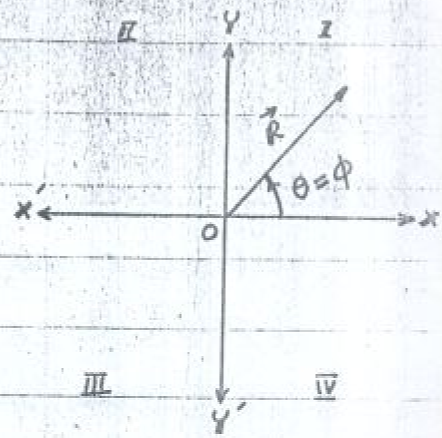
Where θ is the angle make by resultant vector $\vec{R}$ with positive X - axis. The signs of R_x and R_y determine the quadrant in which the resultant vector $\vec{R}$ lies. For that purpose, we proceed as given below:



Find the value of $\tan^{-1}\left(\frac{R_y}{R_x}\right) = \phi$, then angle θ determined as

(a) If both R_x and R_y are positive, then resultant vector $\vec{R}$ lies in the first quadrant and its direction is

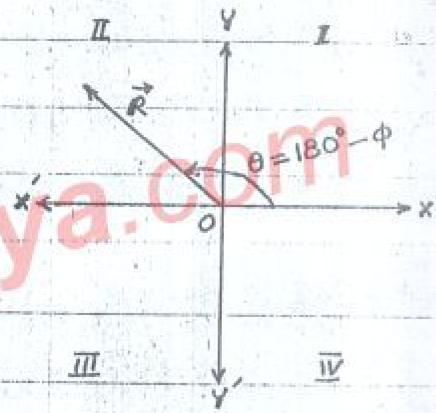
$$\theta = \phi = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$



(b) If R_x is negative and R_y is positive, then resultant vector $\vec{R}$ lies in the second quadrant and its direction is

$$\theta = 180^\circ - \phi$$

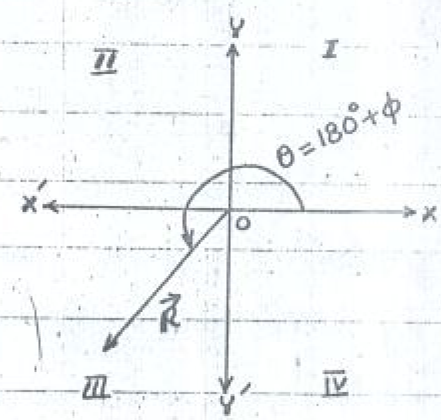
or $$\theta = 180^\circ - \tan^{-1}\left(\frac{R_y}{R_x}\right)$$



(c) If both R_x and R_y are negative, then resultant vector $\vec{R}$ lies in the third quadrant and its direction is

$$\theta = 180^\circ + \phi$$

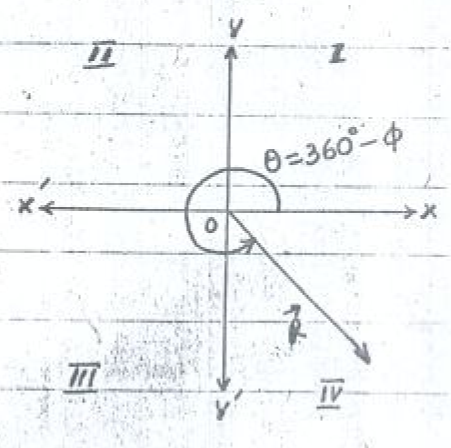
or $$\theta = 180^\circ + \tan^{-1}\left(\frac{R_y}{R_x}\right)$$



(d) If R_x is positive and R_y is negative, then resultant vector $\vec{R}$ lies in the fourth quadrant and its direction is

$$\theta = 360^\circ - \phi$$

or $$\theta = 360^\circ - \tan^{-1}\left(\frac{R_y}{R_x}\right)$$



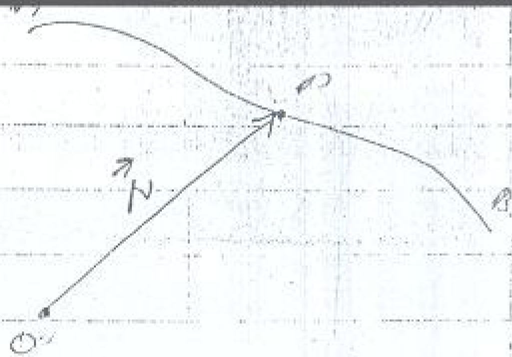
Fsc Physics Chapter 2

POSITION VECTOR.

A vector used to locate the position of a body or a point w.r.t respect to an arbitrary origin, called position vector. It is denoted by $\vec{r}$.

Explanation.

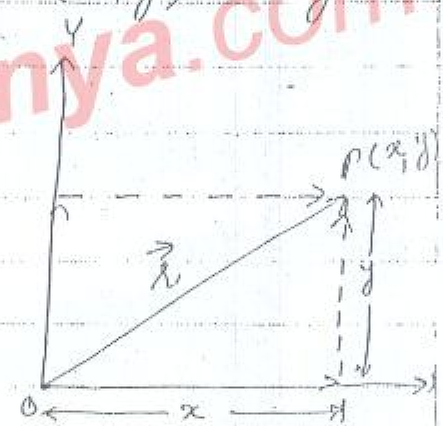
Consider an arbitrary path AB on which a body is moving as shown. Let at any point P, we find the position of the body from origin O.



Then we draw a vector from origin O towards point P, so position vector is

$$\vec{OP} = \vec{r}$$

In case of two dimensions i-e; x-axis and y-axis, the position of point P from O can be represented by coordinates (x, y) as given



$$\vec{OP} = \vec{r} = (x, y)$$

$$\vec{r} = \vec{x} + \vec{y}$$

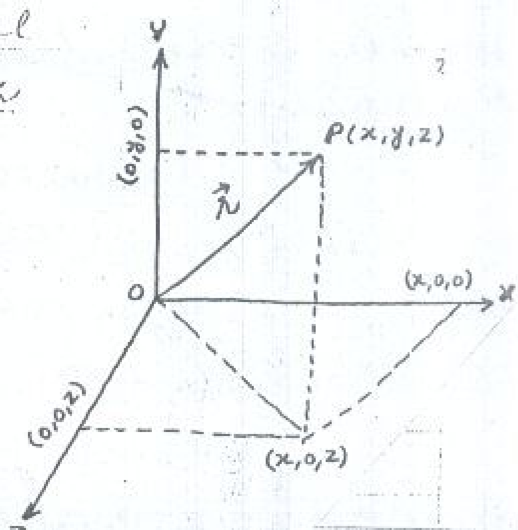
In case of unit vectors

$$\vec{r} = x\hat{i} + y\hat{j}$$

Its magnitude is

$$\boxed{r = \sqrt{x^2 + y^2}}$$

In three dimensional case, the position vector is given as



$$\vec{r} = \vec{x} + \vec{y} + \vec{z}$$

or $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Its magnitude is

$$\boxed{r = \sqrt{x^2 + y^2 + z^2}}$$

Subtraction Of Vectors.

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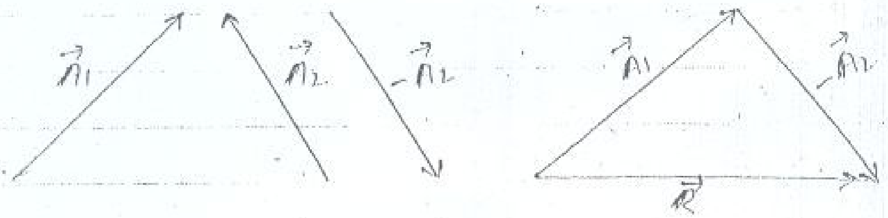
It is same process as addition of vectors with difference is that the vector which is subtracted, first we take its negative and then added in the given vector by Head-to-Tail rule.

For example, vector $\vec{A}_2$ is subtracted from $\vec{A}_1$, then we take negative of $\vec{A}_2$ i-e; $-\vec{A}_2$ and added in $\vec{A}_1$ by head-to-tail rule as shown.



Now, their resultant vector $\vec{R}$ is given as

$$\vec{R} = \vec{A}_1 + (-\vec{A}_2)$$

or $$\boxed{\vec{R} = \vec{A}_1 - \vec{A}_2}$$

~~Addition Of Two Vectors By Rectangular Components~~

~~Consider two vectors $\vec{A}_1$ and $\vec{A}_2$ making angles θ_1 and θ_2 with x-axis respectively as shown and represented by lines $\vec{OA}$ and $\vec{OB}$. These vectors are~~

Product Of Vectors

There are two types of product of vectors given as

(i) Scalar or Dot Product of Two Vectors

(ii) Vector or Cross Product of Two Vectors.

Scalar or Dot Product Of Two Vectors

The product of two vectors which results in a scalar quantity, called scalar product of two vectors. It is also known as dot product because we use dot (•) between two vectors to represent scalar product.

Consider two vectors $\vec{A}$ and $\vec{B}$, then their scalar product can be written as

$$\vec{A} \cdot \vec{B}$$

and read as "$\vec{A}$ dot $\vec{B}$".

The formula of scalar product is

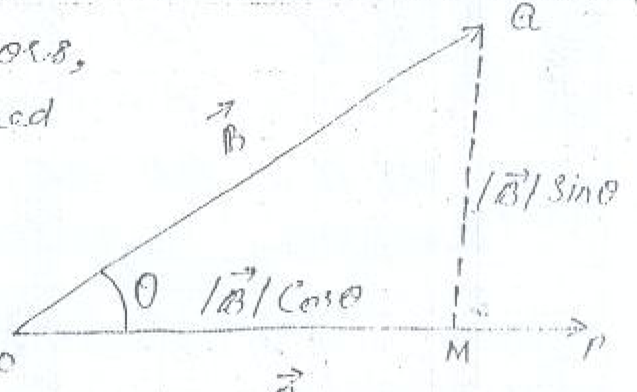
$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

or $$\vec{A} \cdot \vec{B} = AB \cos\theta$$

where θ is the angle between vectors $\vec{A}$ and $\vec{B}$.

EXPLANATION

Consider two vectors $\vec{A}$ and $\vec{B}$ represented by lines $\vec{OP}$ and $\vec{OQ}$ respectively making angle θ with each other as shown. For product of vectors, their tails are joined together as shown.



To multiply $\vec{A}$ with $\vec{B}$ scalarly, draw projection QM from $\vec{B}$ on $\vec{A}$, then their scalar product is

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

or $$\vec{A} \cdot \vec{B} = AB \cos\theta \longrightarrow (1)$$

It can be written as

$\vec{A}\cdot\vec{B}$ = (Magnitude of $\vec{A}$) Times (Scalar projection of $\vec{B}$ on $\vec{A}$)

OR

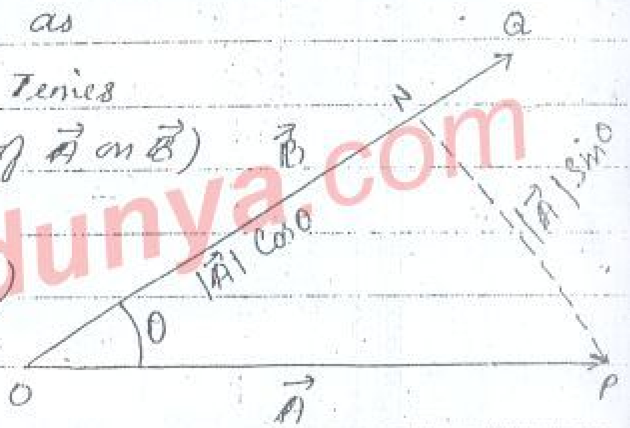
$\vec{A}\cdot\vec{B}$ = (Magnitude of $\vec{A}$) Times (Component of $\vec{B}$ along $\vec{A}$)

Similarly, scalar or dot product of $\vec{B}$ with $\vec{A}$ is given as

$\vec{B}\cdot\vec{A}$ = (Magnitude of $\vec{B}$) Times (Scalar projection of $\vec{A}$ on $\vec{B}$)

OR

$\vec{B}\cdot\vec{A}$ = (Magnitude of $\vec{B}$) Times (Component of $\vec{A}$ along $\vec{B}$)



$\Rightarrow \quad \vec{B}\cdot\vec{A} = |\vec{B}|\,|\vec{A}|\,Cos\theta$

or $\quad \vec{B}\cdot\vec{A} = B\,A\,Cos\theta$

or $\quad \vec{B}\cdot\vec{A} = AB\,Cos\theta \longrightarrow (2)$

Comparing equation (1) and (2)

$$\boxed{\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}}$$

This equation shows that if order of multiplication of scalar or dot product is changed, then scalar or dot product remains same. Hence, scalar or dot product of two vectors is commutative.

EXAMPLES

1. WORK:- It is dot or scalar product of force ($\vec{F}$) and displacement ($\vec{d}$) given as

$$W = \vec{F} \cdot \vec{d} = Fd \cos\theta$$

2. POWER:- It is dot product of force ($\vec{F}$) and velocity ($\vec{V}$) given as

$$P = \vec{F} \cdot \vec{V} = FV \cos\theta$$

3. ELECTRIC FLUX

It is dot product of electric intensity ($\vec{E}$) and vector area ($\vec{A}$) given as

$$\phi_e = \vec{E} \cdot \vec{A} = EA \cos\theta.$$

CHARACTERISTICS OF DOT PRODUCT

1. DISTRIBUTIVE PROPERTY

The dot product of two vectors is distributive. i-e;

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

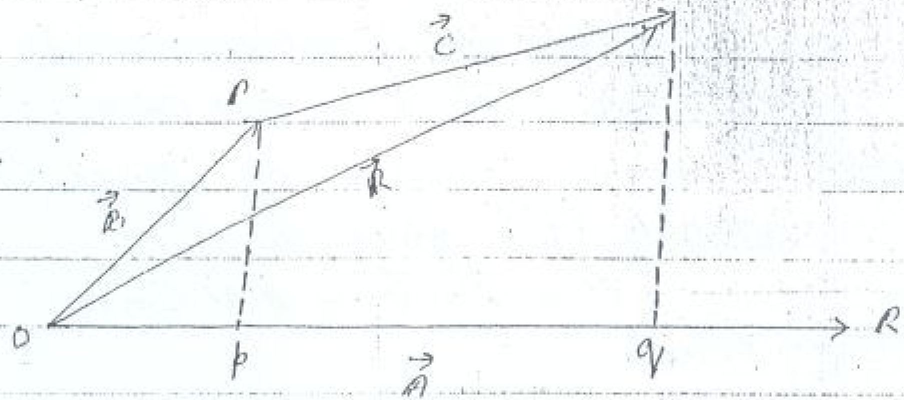
PROOF:- Consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ represented by lines $\vec{OR}$, $\vec{OP}$ and $\vec{PQ}$ respectively as shown in figure. Vectors $\vec{B}$ and $\vec{C}$ are added by head-to-tail rule to get resultant vector $\vec{R}$ as shown. So

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

$$\vec{R} = \vec{B} + \vec{C}$$

According to definition of dot product.

$\vec{A} \cdot (\vec{B} + \vec{C})$ = (Magnitude of $\vec{A}$) times (Scalar projection of ($\vec{B}+\vec{C}$) on $\vec{A}$)



or $\quad \vec{A}\cdot(\vec{B}+\vec{C}) = A|oq| \longrightarrow (1)$

Where

$|oq|$ = Scalar projection of $(\vec{B}+\vec{C})$ on $\vec{A}$

From figure

$$|oq| = |op| + |pq|$$

Put in equation (1)

$$\therefore \vec{A}\cdot(\vec{B}+\vec{C}) = A\left(|op| + |pq|\right)$$

or $\quad \vec{A}\cdot(\vec{B}+\vec{C}) = A|op| + A|pq| \longrightarrow (2)$

Where

$A|op| = A$ (Scalar projection of $\vec{B}$ on $\vec{A}$) $= \vec{A}\cdot\vec{B}$

$A|pq| = A$ (Scalar projection of $\vec{C}$ on $\vec{A}$) $= \vec{A}\cdot\vec{C}$

Put these values in eq. (2)

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

Which is required result.

2. COMMUTATIVE PROPERTY.

The dot product of two vectors is commutative. i-e;

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

PROOF

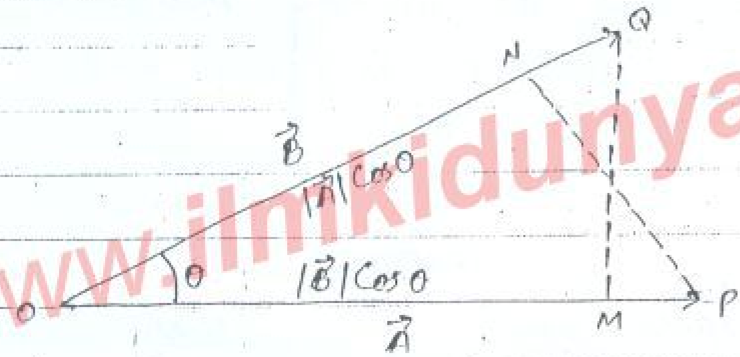
Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other as shown in figure.

The scalar or dot product of $\vec{A}$ with $\vec{B}$ is given as

$\vec{A} \cdot \vec{B}$ = (Magnitude of $\vec{A}$) Times (Scalar projection of $\vec{B}$ on $\vec{A}$)

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

$$\vec{A} \cdot \vec{B} = AB\cos\theta \longrightarrow (1)$$



Similarly, scalar product of $\vec{B}$ with $\vec{A}$ is given as

$\vec{B} \cdot \vec{A}$ = (Magnitude of $\vec{B}$) Times (Scalar projection of $\vec{A}$ on $\vec{B}$)

$$\vec{B} \cdot \vec{A} = |\vec{B}|\,|\vec{A}| \cos\theta$$

$$\vec{B} \cdot \vec{A} = BA\cos\theta$$

$$\vec{B} \cdot \vec{A} = AB\cos\theta \longrightarrow (2)$$

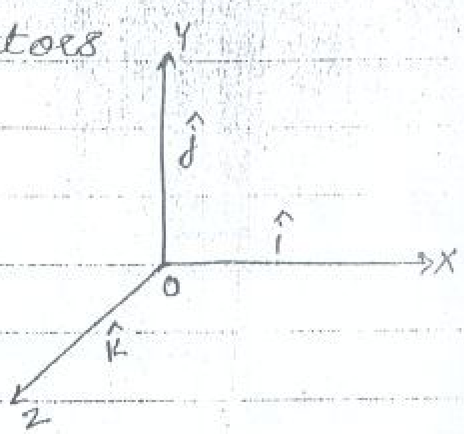
Comparing equation (1) and (2)

$$\boxed{\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}}$$

Which is required result.

In case of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i}$$
$$\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j}$$
$$\hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k}$$



3. Perpendicularity Condition

The dot or scalar product of two perpendicular vectors is zero. i-e;

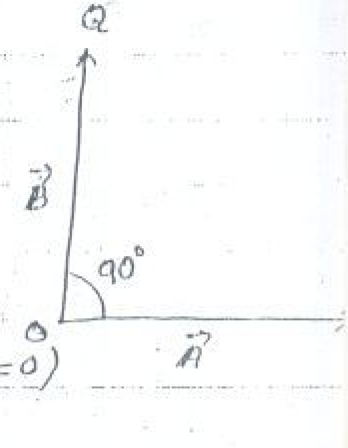
$$\vec{A}\cdot\vec{B} = 0$$

Proof

Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other, then

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

Vectors are said to be perpendicular if angle between them is $90°$ i-e; $\theta = 90°$



$$\therefore \vec{A}\cdot\vec{B} = AB\cos 90°$$

$$\vec{A}\cdot\vec{B} = AB(0) \quad (\because \cos 90° = 0)$$

$$\boxed{\vec{A}\cdot\vec{B} = 0}$$

Which is required result.

In case of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$\hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{i}$$

$$\hat{j}\cdot\hat{k} = 0 = \hat{k}\cdot\hat{j}, \quad \hat{k}\cdot\hat{i} = 0 = \hat{i}\cdot\hat{k}$$

1. Condition Of Equal Vectors or Self Dot Product

The scalar product of two equal vectors is always equal to the square of the magnitude of either vector. i-e;

$$\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{A} = A^2$$

or $$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{B} = B^2$$

Proof: Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other, then

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

The vectors are said to be equal if their magnitude and direction both are same i-e;

$$\vec{A} = \vec{B}$$

if $A = B$ and $\theta = 0^\circ$

$\vec{A}$ ———→

$\vec{B}$ ———→

$\therefore$ $\vec{A}\cdot\vec{B} = AA\cos 0^\circ$

or $\vec{A}\cdot\vec{B} = A^2(1)$ $\qquad(\because \cos 0^\circ = 1)$

$$\boxed{\begin{aligned} \text{or}\quad & \vec{A}\cdot\vec{B} = A^2 \\ \text{or}\quad & \vec{A}\cdot\vec{A} = A^2 \qquad (\because \vec{B} = \vec{A}) \\ \text{or}\quad & \vec{B}\cdot\vec{B} = B^2 \qquad (\because \vec{A} = \vec{B}) \end{aligned}}$$

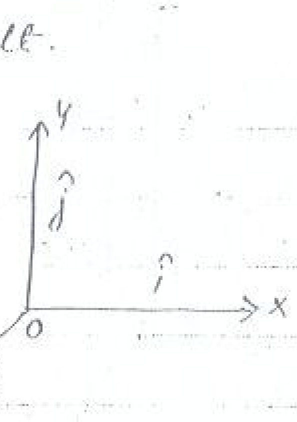
Which is required result.

In case of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$\hat{i}\cdot\hat{i} = 1$$

$$\hat{j}\cdot\hat{j} = 1$$

$$\hat{k}\cdot\hat{k} = 1$$



5. Collinearity Condition

or Dot Product of Parallel or Anti-Parallel Vectors

or Prove that $\vec{A}\cdot\vec{B} = \pm AB$

Proof: Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other, then

$$\vec{A}\cdot\vec{B} = AB\cos\theta \longrightarrow (1)$$

The vectors are said to be collinear if angle between them is $0°$ or $180°$.

Case (i) When $\theta = 0°$, i-e; vectors are parallel.

Put in equation (1)

$\therefore \vec{A}\cdot\vec{B} = AB\cos 0°$

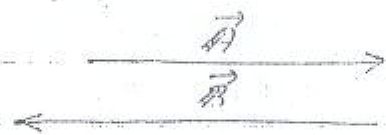
or $\vec{A}\cdot\vec{B} = AB(1)$ $\quad (\because \cos 0° = 1)$

or $\boxed{\vec{A}\cdot\vec{B} = AB} \longrightarrow (2)$

Hence, scalar or dot product of parallel vectors is equal to the product of their magnitudes.

Case (ii) When $\theta = 180°$, i-e; vectors are anti-parallel.

Put in equation (1)



$\therefore \vec{A}\cdot\vec{B} = AB\cos 180°$

or $\vec{A}\cdot\vec{B} = AB(-1)$ $\quad (\because \cos 180° = -1)$

or $\boxed{\vec{A}\cdot\vec{B} = -AB} \longrightarrow (3)$

Hence, scalar or dot product of two anti-parallel vectors is equal to the minus product of their magnitudes.

Combining equation (2) and (3)

$$\therefore \boxed{\vec{A} \cdot \vec{B} = \pm AB}$$

Which is required result.

6. ASSOCIATIVE PROPERTY

The scalar or dot product of two vectors is associative.

Consider, two vectors $\vec{A}$ and $\vec{B}$, and m, n be any two integers, then

$$m\vec{A} \cdot n\vec{B} = mn\vec{A} \cdot \vec{B}$$

$$= \vec{A} \cdot mn\vec{B}$$

$$= n\vec{A} \cdot m\vec{B}$$

$$= mn(\vec{A} \cdot \vec{B})$$

$$= n\vec{B} \cdot m\vec{A}$$

7. DOT PRODUCT OF VECTORS IN RECTANGULAR COMPONENTS

The scalar or dot product of two vectors is equal to the sum of product of magnitudes of corresponding elements. i.e;

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

PROOF: Consider two vectors

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Their dot product is

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

or $\vec{A}\cdot\vec{B} = A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y(\hat{i}\cdot\hat{j}) + A_xB_z(\hat{i}\cdot\hat{k})$
$+ A_yB_x(\hat{j}\cdot\hat{i}) + A_yB_y(\hat{j}\cdot\hat{j}) + A_yB_z(\hat{j}\cdot\hat{k})$
$+ A_zB_x(\hat{k}\cdot\hat{i}) + A_zB_y(\hat{k}\cdot\hat{j}) + A_zB_z(\hat{k}\cdot\hat{k})$

As, we know that

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$
$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = 0$$
$$\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = 0$$
$$\hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = 0$$

$\therefore \vec{A}\cdot\vec{B} = A_xB_x(1) + A_xB_y(0) + A_xB_z(0)$
$+ A_yB_x(0) + A_yB_y(1) + A_yB_z(0)$
$+ A_zB_x(0) + A_zB_y(0) + A_zB_z(1)$

or $\vec{A}\cdot\vec{B} = A_xB_x + 0 + 0 + 0 + A_yB_y$
$+ 0 + 0 + 0 + A_zB_z$

or $$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

Which is required result.

8. <u>ANGLE BETWEEN VECTORS.</u>

To find angle θ between any two vectors $\vec{A}$ and $\vec{B}$, we use dot product as

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

or $$\cos\theta = \vec{A}\cdot\vec{B}/AB$$

or $$\boxed{\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right)}$$

VECTOR OR CROSS PRODUCT OF TWO VECTORS

The product of two vectors which result in a vector quantity, called vector product. It is also known as cross product because to represent vector product, we use cross (×) between two vectors.

Consider two vectors $\vec{A}$ and $\vec{B}$, then their vector or cross product can be written as

$$\vec{A} \times \vec{B}$$

and read as "$\vec{A}$ cross $\vec{B}$"

Its formula is given as

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \hat{n}$$

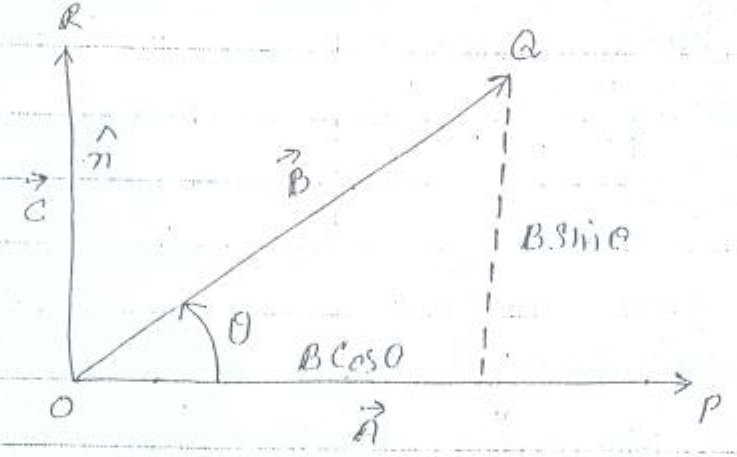
or $$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$.

EXPLANATION

Consider two vectors $\vec{A}$ and $\vec{B}$ represented by lines $\vec{OP}$ and $\vec{OQ}$ making angle θ with each other as shown. To multiply vectors, join tails of the vectors.

For vector product of $\vec{A}$ with $\vec{B}$ to get another vector $\vec{C}$ represented by line $\vec{OR}$, draw perpendicular from $\vec{B}$ on $\vec{A}$ as shown.



Now, the vector product is given as

$$\vec{C} = \vec{A} \times \vec{B}$$

or $\vec{C} = |\vec{A}|\,|\vec{B}|\,\sin\theta \cdot \hat{n}$

or $\vec{C} = AB\sin\theta \cdot \hat{n} \longrightarrow (1)$

Its magnitude is

$$|\vec{C}| = |\vec{A}\times\vec{B}| = AB\sin\theta\,|\hat{n}| = AB\sin\theta\,(1)$$

$$|\vec{C}| = |\vec{A}\times\vec{B}| = AB\sin\theta$$

It can be written as

$$\vec{C} = \vec{A}\times\vec{B} = AB\sin\theta \cdot \hat{n}$$

or $\vec{C} = \vec{A}\times\vec{B} =$ (Magnitude of $\vec{A}$) times (Component of $\vec{B}$ perpendicular to $\vec{A}$) (Unit vector)

where $\hat{n}$ is the unit vector in the direction of $\vec{C}$ and it can be determined by Right Hand Rule.

<u>RIGHT HAND RULE</u>

Stretch right hand with erected thumb, if curl or rotation of fingers shows the vector product of vectors, then thumb shows the direction of resultant vector.

Similarly, vector product of $\vec{B}$ with $\vec{A}$ is given as

$$\vec{B} \times \vec{A} = -\vec{C}$$

or $-\vec{C} = \vec{B}\times\vec{A} =$ (Magnitude of $\vec{B}$) times (Component of A perpendicular to $\vec{B}$) (Unit vector)

or $-\vec{C} = \vec{B}\times\vec{A} = |\vec{B}|\,|\vec{A}|\sin\theta \cdot \hat{n}$

or $-\vec{C} = \vec{B}\times\vec{A} = BA\sin\theta \cdot \hat{n}$

or $-\vec{C} = \vec{B}\times\vec{A} = AB\sin\theta \cdot \hat{n}$

or $\vec{C} = -\vec{B}\times\vec{A} = AB\sin\theta \cdot \hat{n}$ ——→ (2)

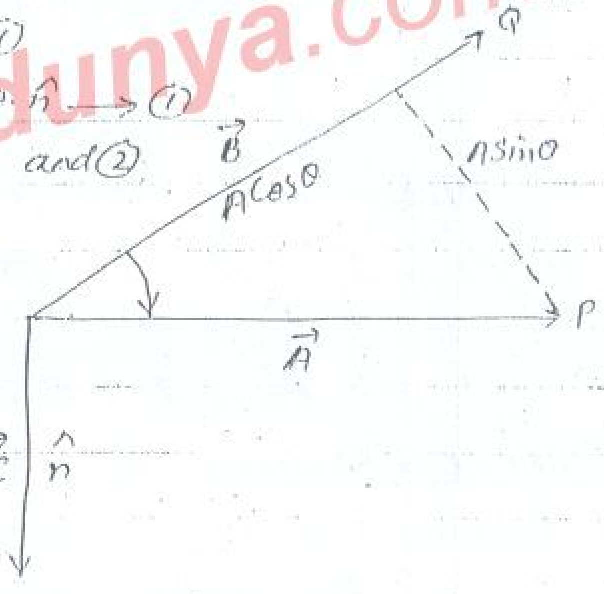
From equation (1)

$\vec{C} = \vec{A}\times\vec{B} = AB\sin\theta \cdot \hat{n}$ ——→ (1)

Comparing equation (1) and (2)

$$\boxed{\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}}$$

$$\Rightarrow \boxed{\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}}$$



Hence, vector or cross product of two vectors is not commutative.

EXAMPLES OF VECTOR PRODUCT

1. TORQUE:- It is cross product of moment arm ($\vec{r}$) and force ($\vec{F}$) given as

Torque $= \vec{\tau} = \vec{r}\times\vec{F}$

or $\vec{\tau} = rF\sin\theta \cdot \hat{n}$

2. ANGULAR MOMENTUM

It is cross product of radius ($\vec{r}$) and linear momentum ($\vec{P}$) given as

$$\vec{L} = \vec{r} \times \vec{P}$$

or $$\vec{L} = rP\sin\theta \cdot \hat{n}$$

3. MAGNETIC FORCE

It is cross product of velocity ($\vec{v}$) and magnetic induction ($\vec{B}$) given as

$$\vec{F}_m = q(\vec{v} \times \vec{B})$$

or $$\vec{F}_m = qvB\sin\theta \cdot \hat{n}$$

Also magnetic force is cross product of length ($\vec{l}$) and magnetic induction ($\vec{B}$) given as

$$\vec{F}_m = I(\vec{l} \times \vec{B})$$

or $$\vec{F}_m = IlB\sin\theta \cdot \hat{n}.$$

CHARACTERISTICS OF VECTOR PRODUCT

1. COMMUTATIVE PROPERTY

The vector or cross product of two vectors is not commutative. i-e;

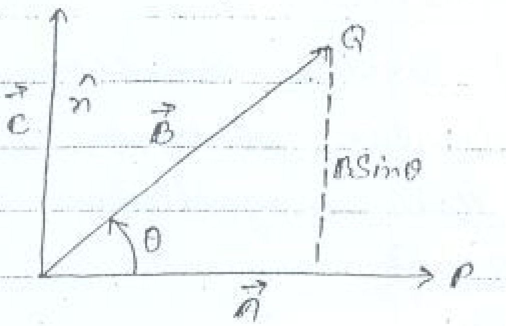
$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

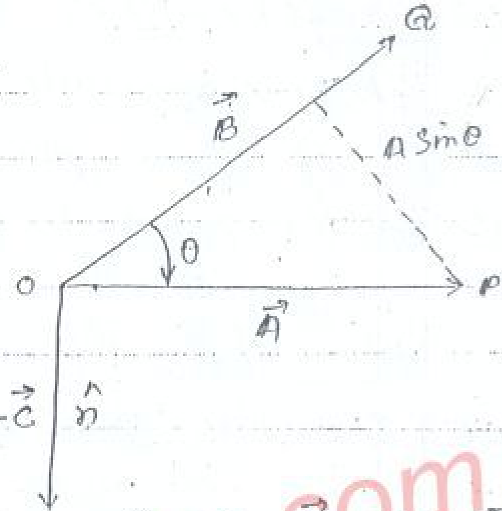
PROOF: Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other, then cross product of $\vec{A}$ with $\vec{B}$ given as

$\vec{C} = \vec{A} \times \vec{B} =$ (Magnitude of $\vec{A}$) Times (Component of $\vec{B}$ perpendicular to $\vec{A}$) (Unit vector)

P 2.40

or $\vec{C} = \vec{A} \times \vec{A} = AB\sin\theta \cdot \hat{n} \longrightarrow (1)$





Similarly, cross product of $\vec{B}$ with $\vec{A}$ is given as

$-\vec{C} = \vec{B} \times \vec{A} =$ (Magnitude of $\vec{B}$) times (Component of $\vec{A}$ perpendicular to $\vec{B}$) (Unit vector)

or $\quad -\vec{C} = \vec{B} \times \vec{A} = BA\sin\theta \cdot \hat{n}$

or $\quad \vec{C} = -\vec{B} \times \vec{A} = AB\sin\theta \cdot \hat{n} \longrightarrow (2)$

Comparing equation (1) and (2)

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\Rightarrow \boxed{\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}}$$

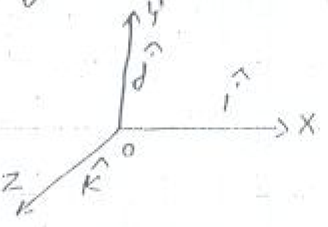
Hence, cross product is not commutative.

In case of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i}$

$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j}$

$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k}$



2. PERPENDICULARITY CONDITION

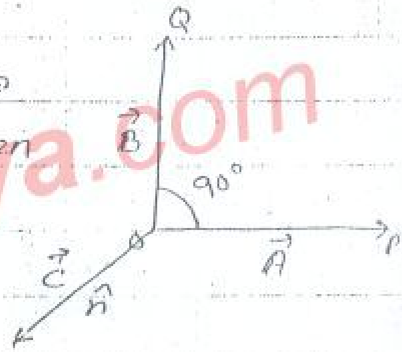
The magnitude of vector or cross product of two perpendicular vectors is equal to the product of their magnitudes. i-e;

$$\vec{A} \times \vec{B} = AB \cdot \hat{n}$$

or $$|\vec{A} \times \vec{B}| = AB$$

PROOF: Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other, then

$$\vec{A} \times \vec{B} = AB\sin\theta \cdot \hat{n}$$



The vectors are said to be perpendicular if angle between them is $90°$. i-e; $\theta = 90°$

$\therefore$ $\vec{A} \times \vec{B} = AB\sin 90° \cdot \hat{n}$

$\vec{A} \times \vec{B} = AB(1) \cdot \hat{n}$

$\because \sin 90° = 1$

$$\therefore \boxed{\vec{A} \times \vec{B} = AB \cdot \hat{n}}$$

Its magnitude is

$$\boxed{|\vec{A} \times \vec{B}| = AB}$$

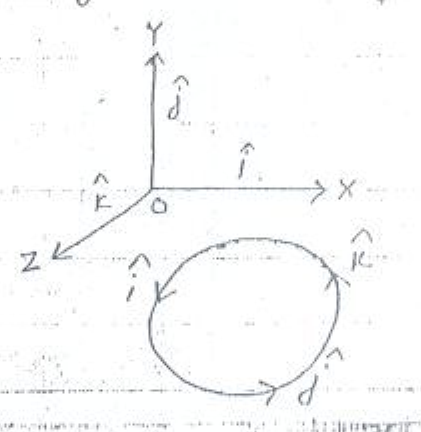
Which is required result.

In case of unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, we have

$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$, $\hat{i} \times \hat{k} = -\hat{j}$



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5. Condition Of Equal Vectors or Self Cross Product

The vector or cross product of equal vectors is zero. i-e;

$$\vec{A} \times \vec{B} = 0$$

or $\vec{A} \times \vec{A} = 0 = \vec{B} \times \vec{B}$

Proof:- Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other, then

$$\vec{A} \times \vec{B} = AB \sin\theta \cdot \hat{n}$$

The vectors are said to be equal if their magnitude and direction both are same i-e;

$$\vec{A} = \vec{B}$$

if $A = B$ and $\theta = 0^\circ$

$\therefore$ $\vec{A} \times \vec{B} = AB \sin 0^\circ \cdot \hat{n}$

or $\vec{A} \times \vec{B} = AB(0) \cdot \hat{n}$ $\quad (\because \sin 0^\circ = 0)$

or $\boxed{\vec{A} \times \vec{B} = 0}$

Also

$\vec{A} \times \vec{A} = 0$ $\quad (\because \vec{B} = \vec{A})$

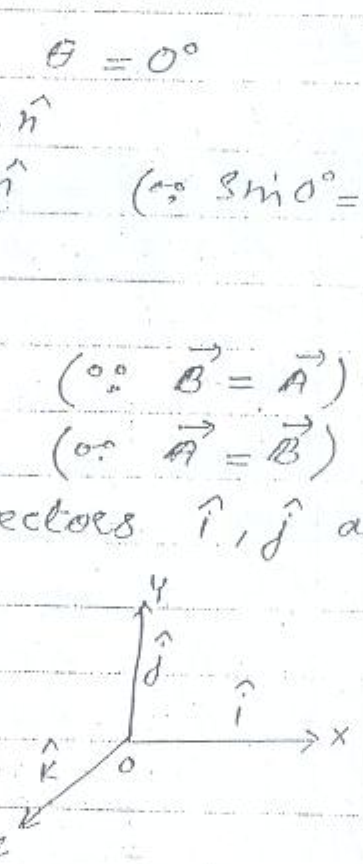
or $\vec{B} \times \vec{B} = 0$ $\quad (\because \vec{A} = \vec{B})$

In case of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$\hat{i} \times \hat{i} = 0$

$\hat{j} \times \hat{j} = 0$

$\hat{k} \times \hat{k} = 0$



4. COLLINEARITY CONDITION

OR CONDITION OF PARALLEL OR ANTI-PARALLEL VECTORS

The vector or cross product of two collinear vectors (Parallel or Anti-Parallel) is always zero. i-e;

$$\vec{A} \times \vec{B} = 0$$

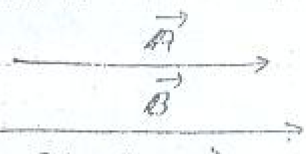
PROOF: Consider two vectors $\vec{A}$ and $\vec{B}$ making angle θ with each other, then

$$\vec{A} \times \vec{B} = AB \sin\theta \cdot \hat{n} \longrightarrow (1)$$

Vectors are said to be collinear if angle between them is 0° or 180°.

Case (i) When $\theta = 0^\circ$ i-e; vectors are parallel.

Put in equation (1)



$\therefore \quad \vec{A} \times \vec{B} = AB \sin 0^\circ \cdot \hat{n}$

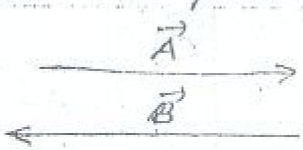
$\vec{A} \times \vec{B} = AB(0) \cdot \hat{n} \quad (\because \sin 0^\circ = 0)$

$$\boxed{\vec{A} \times \vec{B} = 0}$$

Hence, cross product of parallel vectors is zero.

Case (ii) When $\theta = 180^\circ$ i-e; Vectors are anti-parallel.

Put in eq. (1)



$\therefore \quad \vec{A} \times \vec{B} = AB \sin 180^\circ \cdot \hat{n}$

$\vec{A} \times \vec{B} = AB(0) \cdot \hat{n} \quad (\because \sin 180^\circ = 0)$

$$\boxed{\vec{A} \times \vec{B} = 0}$$

Hence, cross product of anti-parallel vectors is zero.

5. DISTRIBUTIVE PROPERTY

The cross or vector product of two vectors is distributive, i-e;

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

6. CROSS PRODUCT IN RECTANGULAR COMPONENTS

The vector or cross product of two vectors in their component form can be written in determinant form as.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

PROOF: Consider two vectors

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$
$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

Their cross product is

$$\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$\vec{A} \times \vec{B} = A_x B_x (\hat{i} \times \hat{i}) + A_x B_y (\hat{i} \times \hat{j}) + A_x B_z (\hat{i} \times \hat{k})$$
$$+ A_y B_x (\hat{j} \times \hat{i}) + A_y B_y (\hat{j} \times \hat{j}) + A_y B_z (\hat{j} \times \hat{k})$$
$$+ A_z B_x (\hat{k} \times \hat{i}) + A_z B_y (\hat{k} \times \hat{j}) + A_z B_z (\hat{k} \times \hat{k})$$

As, we know that

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$
$$\hat{i} \times \hat{j} = \hat{k} \quad , \quad \hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i} \quad , \quad \hat{k} \times \hat{j} = -\hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j} \quad , \quad \hat{i} \times \hat{k} = -\hat{j}$$

Put these values in above equation

$$\therefore \vec{A}\times\vec{B} = A_xB_x(0) + A_xB_y(\hat{k}) + A_xB_z(-\hat{j})$$
$$+ A_yB_x(-\hat{k}) + A_yB_y(0) + A_yB_z(\hat{i})$$
$$+ A_zB_x(\hat{j}) + A_zB_y(-\hat{i}) + A_zB_z(0)$$

or $$\vec{A}\times\vec{B} = 0 + A_yB_z\,\hat{i} - A_zB_y\,\hat{i} - A_xB_z\,\hat{j}$$
$$+ A_zB_x\,\hat{j} + 0 + 0 + A_xB_y\,\hat{k} - A_yB_x\,\hat{k}$$

or $$\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x)$$
$$+ \hat{k}(A_xB_y - A_yB_x)$$

It can be written in determinant form as

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

7. AREA OF PARALLELOGRAM

The magnitude of cross product of two adjacent sides of a parallelogram is equal to the area of parallelogram. i-e;

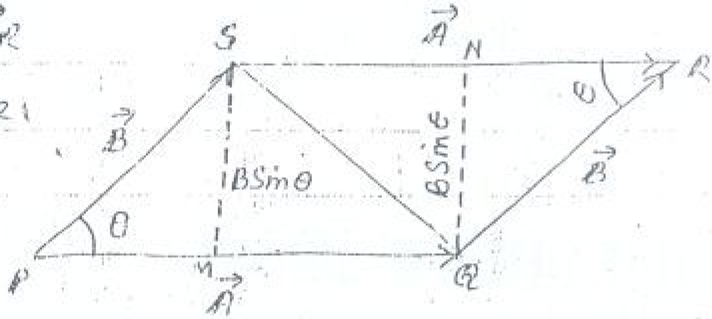
$$\text{Area Of Parallelogram} = |\vec{A}\times\vec{B}|$$

PROOF: Consider a parallelogram PQRS having sides as vectors $\vec{A}$ and $\vec{B}$ as shown.

From figure, we have

$$\vec{PQ} = \vec{A} = \vec{SR}$$
$$\vec{PS} = \vec{B} = \vec{QR}$$

Let θ be the angle between vectors $\vec{A}$ and $\vec{B}$.



Now, cross product of vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A} \times \vec{B} = AB\sin\theta \cdot \hat{n}$$

Its magnitude is

$$|\vec{A} \times \vec{B}| = AB\sin\theta \longrightarrow (1)$$

Now, area of parallelogram PQRS, we have

Area of PQRS = Area of ΔPQS + Area of ΔQRS

As, we know that

Area of triangle = $\frac{1}{2}$ x Base x Height

$\therefore$ Area of PQRS = $\frac{1}{2} AB\sin\theta + \frac{1}{2} AB\sin\theta$

or Area of PQRS = $AB\sin\theta$

or Area of Parallelogram = $AB\sin\theta \longrightarrow (2)$

Comparing equation (1) and (2)

$$\boxed{\text{Area Of Parallelogram} = |\vec{A} \times \vec{B}|}$$

Which is required result.

DIFFERENCE BETWEEN SCALARS AND VECTORS

SCALARS	VECTORS
(1) Scalars are those physical quantities which have magnitude, unit and no direction.	(1) Vectors are those physical quantities which have magnitude, unit and particular direction
(2) Symbolically, scalars are represented by usually capital letters without arrow mark or line i-e; A, B, ------, Z	(2) Symbolically, vectors are always represented by usually capital letters with arrow mark over their head or a line drawn below the letter. i-e; $\vec{A}, \vec{B}, ------, \vec{Z}$ or $\underline{A}, \underline{B}, ------, \underline{Z}$.
(3) Graphically, scalars are represented by either straight, curved or zig-zag lines.	(3) Graphically, vectors are represented by always straight lines with arrow head at one end.
(4) These are added, subtracted, multiplied or divided by simple arithematical methods.	(4) These are added, subtracted, multiplied or divided by special arithematical methods.
(5) Their examples are time, length, density, work, area etc.	(5) Their examples are displacement, velocity, Force, momentum etc.

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TORQUE

The turning effect of a force is called its moment or torque.

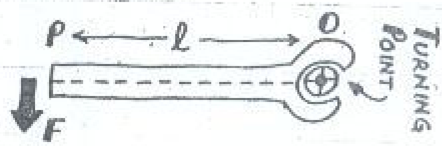
DEFINITION → The product of force and the perpendicular distance from its line of action of force to the point around which the body rotates is called torque. It is denoted by τ.

The perpendicular distance from the line of action of force to the point around which the body rotates is called moment arm.

If the force applied is F and moment arm is l, then the magnitude of torque is

TORQUE = FORCE x MOMENT ARM

$$\boxed{\tau = Fl}$$



ANOTHER DEFINITION

Torque is also defined as "The vector or cross product of moment arm ($\vec{l}$) and force ($\vec{F}$).

So

$$\vec{\tau} = \vec{l} \times \vec{F}$$

or

$$\vec{\tau} = lF\sin\theta \cdot \hat{n}$$

* The direction of torque can be determined by Right Hand Rule.
* The SI-units of torque is N.m.
* The dimension of torque is $[ML^2T^{-2}]$.
* Counter clockwise torque is taken as positive
* Clockwise torque is taken as negative.

Special Cases

Case I: When the line of action of force passes through the turning point, then

Moment Arm $= l = 0$

So

$$\vec{\tau} = \vec{l} \times \vec{F} = lF\sin\theta \cdot \hat{n}$$

$$\vec{\tau} = (0)F\sin\theta \cdot \hat{n} = 0$$

Hence, torque will be minimum.

Case II: When $\theta = 0°$ or $180°$

So

$$\vec{\tau} = \vec{l} \times \vec{F} = lF\sin\theta \cdot \hat{n}$$

If $\theta = 0°$, $\vec{\tau} = lF\sin 0° \cdot \hat{n} = lF(0) \cdot \hat{n}$ $\quad (\because \sin 0° = 0)$

$$\vec{\tau} = 0$$

If $\theta = 180°$, $\vec{\tau} = lF\sin 180° \cdot \hat{n} = lF(0) \cdot \hat{n}$ $\quad (\because \sin 180° = 0)$

$$\vec{\tau} = 0$$

Hence, torque will be minimum in both cases.

Case III: When $\theta = 90°$

So

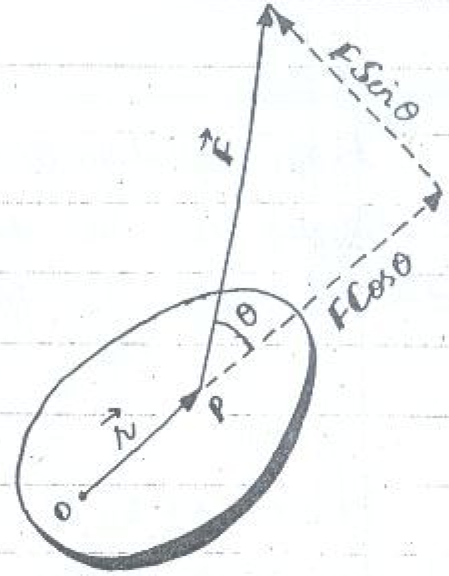
$$\vec{\tau} = \vec{l} \times \vec{F} = lF\sin\theta \cdot \hat{n}$$

$$\vec{\tau} = lF\sin 90° \cdot \hat{n} = lF(1) \cdot \hat{n} \quad (\because \sin 90° = 1)$$

$$\vec{\tau} = lF \cdot \hat{n}$$

Hence, torque will be maximum when force and moment arm are perpendicular to each other.

TORQUE OF A RIGID BODY



Consider the torque due to a force $\vec{F}$ acting on a rigid body. Let the force $\vec{F}$ acts on rigid body at point P whose position vector relative to pivot O is $\vec{r}$. The force $\vec{F}$ resolved into rectangular components as shown given as

(i) $F\cos\theta$, along the direction of $\vec{r}$.

(ii) $F\sin\theta$, perpendicular to the direction of $\vec{r}$.

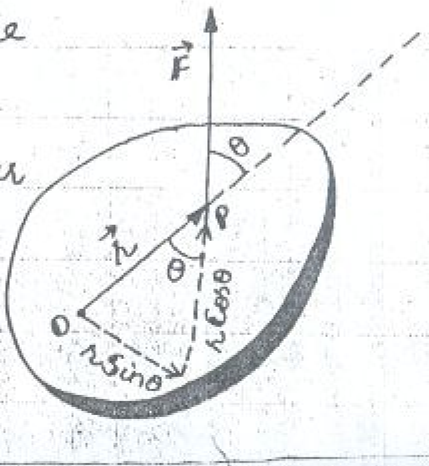
As angle between $\vec{r}$ and $F\cos\theta$ is $0°$, so the torque due to $F\cos\theta$ about O is zero.

Therfore, torque due to $\vec{F}$ is equal to the torque due to its component $F\sin\theta$ only about O. So torque is given as

$$\tau = (F\sin\theta)\, r$$

or $$\tau = rF\sin\theta \longrightarrow ①$$

Where θ is the angle between $\vec{r}$ and $\vec{F}$.



Alternatively, the torque due to $\vec{F}$ about O is equal to component of $\vec{r}$ perpendicular to the line of action of $\vec{F}$ as shown in figure, given as

$$\tau = (r\sin\theta)\, F$$

or $$\tau = rF\sin\theta \longrightarrow ②$$

From equation ① and ②, it can be seen that the torque can be defined by vector product of position vector $\vec{r}$ and the force $\vec{F}$ given as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\text{or} \quad \vec{\tau} = rF\sin\theta \cdot \hat{n} \longrightarrow ③$$

Where $(rF\sin\theta)$ is the magnitude of the torque and the unit vector $\hat{n}$ represents the direction of the torque which is perpendicular to the plane containing vectors $\vec{r}$ and $\vec{F}$.

As force determines the linear acceleration in a body whereas torque acting on a body determines its angular acceleration. Hence, torque is counter part of force for rotational motion.

EQUILIBRIUM

"If a body is at rest or move with uniform velocity under the action of a number of forces, then the body is said to be in equilibrium."

TYPES OF EQUILIBRIUM

There are two types of equilibrium

1. Static Equilibrium
2. Dynamic or Kinetic Equilibrium

STATIC EQUILIBRIUM

If a body is at rest under the action of a number of forces, then body is said to be in static equilibrium.

For example

* A book lying on a table.
* A man holding a weight.

DYNAMIC EQUILIBRIUM

If a body is moving with uniform velocity under the action of a number of forces, then body is said to be in dynamic or kinetic equilibrium.

For example

* A car moving with uniform velocity.
* The jumping of a paratrooper.

CONDITIONS OF EQUILIBRIUM

There are two conditions of equilibrium

FIRST CONDITION OF EQUILIBRIUM OR EQUILIBRIUM OF FORCES

STATEMENT: The vector sum of all the forces acting on a body must be equal to zero.

Mathematically, using the symbol $\sum \vec{F}$ for the sum of all the forces, we can write

$$\boxed{\sum \vec{F} = 0} \longrightarrow (1)$$

In case of coplanar forces, the first condition of equilibrium is usually expressed in terms of X and Y-components of the forces. Hence, the sum of X-components of all the forces is equal to zero. Similarly, the sum of Y-components of all the forces is equal to zero. Mathematically

$$\boxed{\sum \vec{F}_x = 0} \longrightarrow (2)$$

and

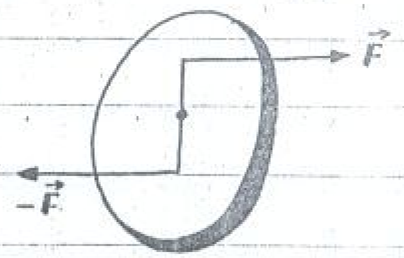
$$\boxed{\sum \vec{F}_y = 0} \longrightarrow (3)$$

It may be noted that if rightward forces are taken as positive then leftward forces are taken as negative. Similarly, if upward forces are taken as positive then downward forces are taken as negative.

SECOND CONDITION OF EQUILIBRIUM OR EQUILIBRIUM OF TORQUES

STATEMENT: The vector sum of all the torques acting on a body about any arbitrary axis should be equal to zero.

In order to understand second condition of equilibrium, consider two equal and opposite forces are acting on a rigid body as shown in figure. Although the first condition of equilibrium is satisfied i-e; $\sum \vec{F} = 0$ but body may rotate in the clock-wise direction under the action of these two forces.



So, the body have clock-wise torques and it will be in equilibrium if sum of torques must be zero. This is second condition of equilibrium. Mathematically, it is written as

$$\boxed{\sum \vec{\tau} = 0} \longrightarrow (4)$$

Fsc Physics Chapter 2

TRANSLATIONAL EQUILIBRIUM

When first condition of equilibrium is satisfied i-e; $\sum \vec{F} = 0$, then there is no linear acceleration and body will be in translational equilibrium.

ROTATIONAL EQUILIBRIUM

When second condition of equilibrium is satisfied i-e; $\sum \vec{\tau} = 0$, then there is no angular acceleration and body will be in rotational equilibrium.

COMPLETE EQUILIBRIUM

When both the conditions of equilibrium are satisfied i-e;

First condition $\rightarrow \sum \vec{F} = 0 \Rightarrow \sum \vec{F}_x = 0$ and $\sum \vec{F}_y = 0$

Second condition $\rightarrow \sum \vec{\tau} = 0$

Then body will be in complete equilibrium.

Fsc Physics Chapter 2

SHORT QUESTIONS & ANSWERS

2.1 Define the terms

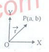
(i) Unit vector (ii) Position vector and (iii) Components of a vector.

Ans:

(i) **Unit vector:** A vector whose magnitude is one is called unit vector. A unit vector $\hat{A}$ in the direction of vector $\vec{A}$ is obtained by dividing the vector $\vec{A}$ by its magnitude A. Thus

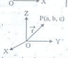
$$\hat{A} = \frac{\vec{A}}{A}$$

(ii) **Position vector:** The position vector is a vector that describes the location of a particle with respect to the origin.



The position vector $\vec{r}$ of a point P(a, b) in xy plane is

$$\vec{r} = a\hat{i} + b\hat{j}$$



and the position vector r of a point P(a, b, c) in space is

$$\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$$

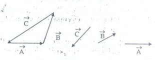
(iii) **Component of a vector:** A component of a vector is its effective value in a given direction. A vector can be considered as the resultant of its component vectors along the specified directions.

2.2 The vector sum of three vectors gives a zero resultant. What can be the orientation of the vectors?

Ans: The vector sum of three vector can be zero if they can be drawn in such a way that they make a closed triangle.

Example:

Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors. Their resultant vector can be find by head to tail rule. From figure it is evident that

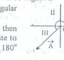


$$\vec{R} = \vec{A} + \vec{B} + \vec{C} = 0$$

2.3 Vector A lies in the xy-plane. For what orientation will both of its rectangular components be negative? For what orientation will its components have opposite signs?

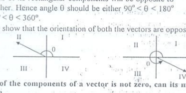
Ans: If a vector A lies in the 3rd quadrant i.e., it makes an angle θ such that $180° < \theta < 270°$ then both its rectangular components will be negative.

Figure shows that the orientation of both rectangular components of vector are negative.



If a vector A lies in the 2nd quadrant or 4th quadrant then the signs of its rectangular components will be opposite to each other. Hence angle θ should be either $90° < \theta < 180°$ or $270° < \theta < 360°$.

Figures show that the orientation of both the vectors are opposite to each other.



2.4 If one of the components of a vector is not zero, can its magnitude be zero? Explain.

Ans: No, if one of the components of a vector is not zero but other components are zero, then the magnitude of resultant vector cannot be zero.

In case of a null vector, its component vectors are equal in magnitude but opposite in direction.

2.5 Can a vector have a component greater than the vector's magnitude?

Ans: No, a vector cannot have a component which has magnitude greater than the magnitude of the original vector. It can have a component of same magnitude when the component in any other specified direction has zero magnitude.

2.6 Can the magnitude of a vector have a negative value?

Ans: No, the magnitude of a vector always has a positive value.

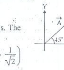
2.7 If $\vec{A} + \vec{B} = 0$, what can you say about the components of the two vectors?

Ans: If $\vec{A} + \vec{B} = 0$, then the sum of their respective components will also be zero, i.e., $A_x + B_x = 0$ and $A_y + B_y = 0$.

2.8 Under what circumstances would a vector have components that are equal in magnitude?

Ans: In case of rectangular components, the magnitudes of the two components will be equal if the vector makes an angle of 45° with +ve x-axis. (Fig.)

Example:



Let vector $\vec{A}$ makes an angle of 45° with +ve x-axis. The magnitude of its rectangular components are

$$A_x = A\cos 45° = \frac{A}{\sqrt{2}} \qquad \left(\because \cos 45° = \frac{1}{\sqrt{2}}\right)$$

$$\text{and} \quad A_y = A\sin 45° = \frac{A}{\sqrt{2}} \qquad \left(\because \sin 45° = \frac{1}{\sqrt{2}}\right)$$

$$\therefore \quad A_x = A_y$$

2.9 Is it possible to add a vector quantity to a scalar quantity? Explain.

Ans: No, only those physical quantities can be added which are alike. Different physical quantities cannot be added. A vector quantity and scalar quantity being different cannot be added.

2.10 Can you add zero to a null vector?

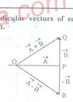
Ans: No, zero cannot be added to null vector, as one i.e; zero is scalar and other is a vector.

2.11 Two vectors have unequal magnitudes. Can their sum be zero? Explain.

Ans: No, the sum of two unequal vectors cannot be zero. What ever, may be their directions, there will be a resultant vector whose magnitude will be in between the difference of their magnitudes or sum of their magnitude.

2.12 Show that the sum and difference of two perpendicular vectors of equal lengths are also perpendicular and of the same length.

Ans: Let there are two vectors A and B of equal length. Using head to tail rule, their sum and difference is shown in the figure. It is seen that their resultants $\overrightarrow{OQ}$ and $\overrightarrow{OQ}$ are also perpendicular and of the same length.



2.13 How would the two vectors of the same magnitude have to be oriented, if they were to be combined to give a resultant equal to a vector of the same magnitude?

Ans: If the angle between two vectors of same magnitude is 120°, the magnitude of their resultant vector is also the same.

This can be proved as follows:

Let $\vec{F}_1$ and $\vec{F}_2$ are two vectors whose magnitudes are same, say F their resultant can be found by using the formula;

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

As $F_1 = F_2 = F$

$$\therefore \quad R = \sqrt{F^2 + F^2 + 2F.F\cos\theta}$$

$$R = \sqrt{2F^2 + 2F^2\cos\theta}$$

If θ = 120° then

$$R = \sqrt{2F^2 + 2F^2\cos 120°}$$

$$= \sqrt{2F^2 + 2F^2\left(-\frac{1}{2}\right)}$$

$$= \sqrt{2F^2 - F^2} \qquad \left(\because \cos 120° = -\frac{1}{2}\right)$$

$$R = \sqrt{F^2} = F$$

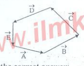
2.14 The two vectors to be combined have magnitudes 60 N and 35 N. Pick the correct answer from those given below and tell why is it the only one of the three that is correct.

(i) 100 N (ii) 70 N (iii) 20 N

Ans: The minimum sum of two given vectors is equal to the difference of their magnitudes which is 25 N and maximum sum is equal to the sum of their magnitudes which is 95 N. Hence, the sum cannot be less than 25 N and more than 95 N. Thus the only correct answer is (ii) which is 70 N.

5 Suppose the sides of a closed polygon represent vector arranged head to tail. What is the sum of these vectors?

ns: If vectors are arranged head to tail and they make a closed polygon, then their resultant sum is zero.



$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E} = 0$$

2.16 Identify the correct answer:

(i) Two ships X and Y are traveling in different directions at equal speeds. The actual direction of motion of X is due north but to an observer on Y, the apparent direction of motion of X is north-east. The actual direction of motion of Y as observed from the ship will be

(A) East (B) west (C) south-east (D) south-west

(ii) A horizontal force F is applied to a small object P of mass m at rest on a smooth plane inclined at an angle θ to the horizontal as shown in Fig. 2.22. The magnitude of the resultant force acting up and along the surface of the plane, on the object is

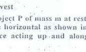
(a) F cos θ - mg sin θ

(b) F sin θ - mg cos θ

(c) F cos θ + mg cos θ

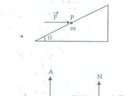
(d) F sin θ + mg sin θ

(e) mg tan θ



Ans:

Let A bet the position of ship X and B be the position of ship Y. From B (ship Y), the position of A (ship X) is north-east. Hence, from the figure it is clear that ship B is heading towards west.



(i) The correct answer is (B) West.

(ii) The correct answer is (a) F cos θ - mg sin θ

.17 If all the components of the vectors, $\vec{A}_1$ and $\vec{A}_2$ were reversed, how would this alter $\vec{A}_1 \times \vec{A}_2$?

ns: If all the components of the vectors $\vec{A}_1$ and $\vec{A}_2$ were reversed, there will be no effect on the resultant of $\vec{A}_1 \times \vec{A}_2$.

2.18 Name the three different conditions that could make $\vec{A}_1 \times \vec{A}_2 = 0$.

Ans: $A_1 \times A_2 = 0$ can be zero if:

(i) A_1 is a null vector or

(ii) A_2 is a null vector or

(iii) A_1 and A_2 are parallel or anti-parallel vectors, for them

$$A_1 \times A_2 = A_1A_2\sin 0° = A_1A_2\sin 180° = 0$$

2.19 Identify true or false statements and explain the reason.

(a) A body in equilibrium implies that it is not moving nor rotating.

(b) If coplanar forces acting on a body form a closed polygon, and then th body is said to be in equilibrium.

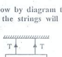
Ans:

(a) This is a false statement since a body may be in equilibrium if it is moving wi uniform velocity.

(b) This is a correct statement since a closed polygon represents zero vector sum forces and this it the 1st condition of equilibrium.

2.20 A picture is suspended from a wall by two strings. Show by diagram tl configuration of the strings for which the tension in the strings will l minimum.

Ans: The tension in the suspension strings will be minimum for the configuration shown in the figure. In this case, the tension in each string will be half that of weight.



$$2T = \omega \quad \text{or} \quad T = \frac{\omega}{2}$$

2.21 Can a body rotate about its centre of gravity under the action of its weight?

Ans: No, a body cannot rotate about its centre of gravity under the action of its wei since the moment arm will be zero and hence, turning effect or torque will zero.

IMPORTANT FORMULAE

★ Unit vector $= \hat{A} = \dfrac{\vec{A}}{A}$

★ Magnitude of position $= r = \sqrt{a^2 + b^2}$ (For two dimensional plane)

$r = \sqrt{a^2 + b^2 + c^2}$ (For three dimensional plane)

★ $A = \sqrt{A_x^2 + A_y^2}$