3.11. System of Linear Equations

Consider a system of three linear equations in three variables

$$a_1x+b_1y+c_1z=k_1$$
$$a_2x+b_2y+c_2z=k_2$$
$$a_3x+b_3y+c_3z=k_3$$

Homogenous Sys.

$(k_1=k_2=k_3=0)$

Trivial / Unique Sol.

$|A|\neq 0$

Non – Trivial / Infinite Sol.

$|A|=0$

Non – Homogenous Sys.

(atleast one of $k_i (i=1,2,3)$ is non – zero)

Solution Exist

Rank $A:b$ = Rank A

finite Solution

Rank = No. of equations

infinite Solution

Rank ≠ No. of equations

Solution Not Exist

Rank $A:b$ ≠ Rank A

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

A system of linear equations is said to be **consistent** if the system has a unique solution or it has infinitely many solution.

A system of linear equations is said to be **inconsistent** if the system has no solution.

3.12. CRAMER'S RULE

Consider a system of equations

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

We write the above system in matrix form

AX = B ....(i) where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

If $|A| \neq 0$ then x,y,z can be calculated as:

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{|A|}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{|A|} \quad \text{and} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{|A|}$$

This method of solving the system is called Cramer's Rule.

Example 3: Use Crammer's rule to solve the system. $\left.\begin{aligned} 3x_1 + x_2 - x_3 &= -4 \\ x_1 + x_2 - 2x_3 &= -4 \\ -x_1 + 2x_2 - x_3 &= 1 \end{aligned}\right\}$ ***(T.B Page No.137)***

Solution: Here $|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3(-1+4) - 1(-1-2) - 1(2+1)$

$= 9 + 3 - 3 = 9$

So $x_1 = \dfrac{\begin{vmatrix} -4 & 1 & -1 \\ -4 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}}{9} = \dfrac{-4(-1+4) - 1(4+2) - 1(-8-1)}{9}$

$= \dfrac{-12-6+9}{9} = \dfrac{-9}{9} = -1$

$$x_2 = \frac{\begin{vmatrix} 3 & -4 & -1 \\ 1 & -4 & -2 \\ -1 & 1 & -1 \end{vmatrix}}{9} = \frac{3(4+2)+4(-1-2)-1(1-4)}{9}$$

$$= \frac{18-12+3}{9} = \frac{9}{9} = 1$$

$$x_3 = \frac{\begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ -1 & 2 & 1 \end{vmatrix}}{9} = \frac{3(1+8)-1(1-4)-4(2+1)}{9}$$

$$= \frac{27+3-12}{9} = \frac{18}{9} = 2$$

Hence $x_1 = -1$, $x_2 = 1$, $x_3 = 2$

EXERCISE 3.5

Q1. **Solve the following systems of linear equations by Cramer's Rule:** (*T.B Page No. 138*)

(i)
$$\begin{aligned} 2x+2y+z&=3 \\ 3x-2y-2z&=1 \\ 3x-2y-2z&=1 \end{aligned}$$

(iii)
$$\begin{aligned} 2x_1-x_2+x_3&=8 \\ x_1+2x_2+2x_3&=6 \\ x_1-2x_2-x_3&=1 \end{aligned}$$

Solution: The matrix form of system of equations is

$$\underset{A}{\begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}$$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$$

$$= 2(6+2) - 2(-9+10) + 1(3+10)$$

$$= 2(8) - 2(1) + 1\,(13)$$

$$= 16 - 2 + 13 = 27 \neq 0$$

$$x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{27} = \frac{3\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}}{27}$$

$$= \frac{3(6+2)-2(-3+4)+1(1+4)}{27} = \frac{3(8)-2(1)+1(5)}{27} = \frac{24-2+5}{27} = \frac{27}{27} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{27} = \frac{2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{27}$$

$$= \frac{2(-3+4)-3(-9+10)+1(6-5)}{27} = \frac{2(1)-3(1)+1(1)}{27} = \frac{2-3+1}{27} = \frac{0}{27} = 0$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{27} = \frac{2\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}}{27}$$

$$= \frac{2(-4-1)-2(6-5)+3(3+10)}{27} = \frac{2(-5)-2(1)+3(13)}{27} = \frac{-10-2+39}{27} = \frac{27}{27} = 1$$

Hence $x = 1, y = 0, z = 1$

(iii) The matrix form of system of equations is

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

$AX = B$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(-2+4) + 1(-1-2) + 1(-2-2) = 2(2) + 1(-3) + 1(-4)$$

$$= 4 - 3 - 4 = -3 \neq 0$$

$$x_1 = \frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{-3} = \frac{8\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 6 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 6 & 2 \\ 1 & -2 \end{vmatrix}}{-3}$$

$$= \frac{8(-2+4)+1(-6-2)+1(-12-2)}{-3} = \frac{8(2)+1(-8)+1(-14)}{-3}$$

$$= \frac{16-8-14}{-3} = \frac{-6}{-3} = 2$$

$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{-3} = \frac{2\begin{vmatrix} 2 & 6 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 6 \\ 1 & 1 \end{vmatrix} + 8\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}}{-3}$$

$$= \frac{2(-6-2)-8(-1-2)+1(1-6)}{-3} = \frac{2(14)+1(-5)+8(-4)}{-3}$$

$$= \frac{-16+24-5}{-3} = \frac{3}{-3} = -1$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{-3}$$

$$= \frac{2(2+12)+1(1-6)+8(-2-2)}{-3}$$

$$= \frac{28-5-32}{-3} = \frac{-9}{-3} = 3$$

Hence $x_1 = 2, x_2 = -1, x_3 = 3.$

Q2. Use matrices to solve the following systems:

$$2x_1 + x_2 + 3x_3 = 3$$

(ii) $x_1 + x_2 - 2x_3 = 0$

$$-3x_1 - x_2 + 2x_3 = -4$$

Solution: The given system of linear equation can be written in matrix form as

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$AX = B$$

Where $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$

$$X = A^{-1}B \qquad \ldots\ldots\ldots \text{(i)}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix}$$

Expand by R_1

$$|A| = 2\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix}$$

$$= 2(2-2) - 1(2-6) + 3(-1+3)$$

$$= 2(0) - 1(-4) + 3(2)$$

$$= 0 + 4 + 6 = 10 \neq 0$$

$$(\text{Co-factor of } A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = (-1)^2(2-2) = 0$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = (-1)^3(2-6) = (-1)(-4) = 4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = (-1)^4(-1+3) = 2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = (-1)^3(2+3) = -5$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} = (-1)^4(4+9) = 13$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = (-1)^5(-2+3) = -1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} = (-1)^4(-2-3) = -5$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = (-1)^5(-4-3) = (-1)(-7) = 7$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^6(2-1) = 1$$

$$\text{Co-factor of } A = \begin{bmatrix} 0 & 4 & 2 \\ -5 & 13 & -1 \\ -5 & 7 & 1 \end{bmatrix}$$

$$\text{adj}(A) = (\text{co-factor})^t = \begin{bmatrix} 0 & 4 & 2 \\ -5 & 13 & -1 \\ -5 & 7 & 1 \end{bmatrix}^t$$

$$\text{adj}A = \begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}$$

Put the value of X, A^{-1} and B in eq. (i)

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0+0+20 \\ 12+0+(-28) \\ 6+0-4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -16 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{20}{10} \\ -\frac{16}{10} \\ \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 2 \\ -8/5 \\ 1/5 \end{bmatrix}$$

Hence $x_1 = 2, x_2 = \frac{-8}{5}, x_3 = \frac{1}{5}$

Q3. **Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms .**

(ii)
$$\begin{aligned} x+2y+z&=2 \\ 2x+y+2z&=-1 \\ 2x+3y-z&=9 \end{aligned}$$

Solution: The augmented matrix of the given system is

$$\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & -1 \\ 2 & 3 & -1 & : & 9 \end{bmatrix}$$

We reduced the above matrix by applying row operations.

$$\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & -1 \\ 2 & 3 & -1 & : & 9 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -3 & 0 & : & -5 \\ 0 & -1 & -3 & : & 5 \end{bmatrix} \begin{matrix} R_2+(-2)R_1 \\ R_3+(-2)R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & 5/3 \\ 0 & -1 & -3 & : & 5 \end{bmatrix} \quad \text{by } \left(-\frac{1}{3}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & -3 & : & \frac{20}{3} \end{bmatrix} \quad \text{by } R_3 + R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \left(-\frac{1}{3}\right)R_3 \qquad \text{.......(A)}$$

Which is required echelon form

$$x+2y+z=2 \qquad \text{....... (i)}$$

$$y=\frac{5}{3} \qquad \text{....... (ii)}$$

$$z=\frac{-20}{9} \qquad \text{....... (ii)}$$

Putting the value of y and z in eq. (i)

$$x+2y+z=2$$

$$x+2\left(\frac{5}{3}\right)-\frac{20}{9}=2$$

$$x+\frac{10}{3}-\frac{20}{9}=2$$

$$x=2-\frac{10}{3}+\frac{20}{9}$$

$$x=\frac{18-30+20}{9}=\frac{8}{9}$$

Thus $\quad x=\frac{8}{9}, y=\frac{5}{3}, z=\frac{-20}{9}$

By Reduced Echelon Method

From (A)

$$\Rightarrow \underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 & : & -\frac{4}{3} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} R_1 - 2R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 & : & \frac{8}{9} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} R_1 - R_3$$

$$\Rightarrow x = \frac{8}{9}, y = \frac{5}{3}, z = -\frac{20}{9}$$

Q4. Solve the following systems of homogeneous linear equations:

(ii) $x_1 + 4x_2 + 2x_3 = 0$
$2x_1 + x_2 - 3x_3 = 0$
$3x_1 + 2x_2 - 4x_3 = 0$

Solution: The given system of equation is

$$x_1 + 4x_2 + 2x_3 = 0 \quad \text{......... (i)}$$
$$2x_1 + x_2 - 3x_3 = 0 \quad \text{......... (ii)}$$
$$3x_1 + 2x_2 - 4x_3 = 0 \quad \text{......... (iii)}$$

Let the matrix of coefficients is

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{vmatrix} = 1\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(-4+6) - 4(-8+9) + 2(4-3) = 2 - 4(1) + 2(1)$$
$$= 2 - 4 + 2 = 0$$

Since $|A|=0$ so the system has infinite non-trivial solution.

Solving first two equation (i) and (ii) of the system we have

Adding eq. (i) and (ii) we get

$$x_1 + 4x_2 + 2x_3 = 0$$
$$2x_1 + x_2 - 3x_3 = 0$$
$$\overline{3x_1 + 5x_2 - x_3 = 0} \quad \text{......... (iv)}$$

Subtracting eq. (ii) from eq. (i)

$$x_1 + 4x_2 + 2x_3 = 0$$
$$\pm 2x_1 \pm x_2 \mp 3x_3 = 0$$

$$-x_1 + 3x_2 + 5x_3 = 0$$
$$\Rightarrow \quad x_1 = 3x_2 + 5x_3 \quad \ldots\ldots\ldots (v)$$

Put the value of x_1 in eq. (iv)

$$3(3x_2 + 5x_3) + 5x_2 - x_3 = 0$$
$$9x_2 + 15x_3 + 5x_2 - x_3 = 0$$
$$14x_2 + 14x_3 = 0$$
$$x_2 + x_3 = 0$$
$$\Rightarrow \quad \boxed{x_2 = -x_3}$$

Put the value of x_2 in eq. (v)

$$x_1 = 3(-x_3) + 5x_3$$
$$\Rightarrow \quad \boxed{x_1 = 2x_3}$$

Putting $x_2 = -x_3$ and $x_1 = 2x_3$ in eq. (iii) we see that

$$3(2x_3) + 2(-x_3) - 4x_3 = 0$$
$$6x_3 - 2x_3 - 4x_3 = 0$$

This shows that eq. (i), (ii) and (iii) are satisfying by

$x_1 = 2t$, $x_2 = -t$, $x_3 = t$ for any real value of t.

Thus system is consistent and eq. (i), (ii) and (iii) has infinitely many solutions.

Q5. Find the value of λ for which the following systems have a non-trivial solution. Also solve the system for the value of λ.

(ii)
$$x_1 + 4x_2 + \lambda x_3 = 0$$
$$2x_1 + x_2 - 3x_3 = 0$$
$$3x_1 + \lambda x_2 - 4x_3 = 0$$

Solution: Let A be the matrix of the coefficients

$$x_1 + 4x_2 + \lambda x_3 = 0$$
$$2x_1 + x_2 - 3x_3 = 0$$
$$3x_1 + \lambda x_2 - 4x_3 = 0$$

Let
$$A = \begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix}$$

$$\therefore \quad |A| = 0$$

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$$

Expand by R_1

$$1\begin{vmatrix} 1 & -3 \\ \lambda & -4 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix} + \lambda\begin{vmatrix} 2 & 1 \\ 3 & \lambda \end{vmatrix} = 0$$

$$1(-4 + 3\lambda) - 4(-8 + 9) + \lambda(2\lambda - 3) = 0$$
$$-4 + 3\lambda - 4 + 2\lambda^2 - 3\lambda = 0$$

$$2\lambda^2 - 8 = 0$$

$$\Rightarrow \quad 2\lambda^2 = 8$$

$$\Rightarrow \quad \lambda^2 = 4$$

$$\Rightarrow \quad \lambda = \pm 2$$ Taking square root on both sides

For $\lambda = 2$:

The given system is

$x_1 + 4x_2 + 2x_3 = 0$ .... (i)

$2x_1 + x_2 - 3x_3 = 0$ .... (ii)

$3x_1 + 2x_2 - 4x_3 = 0$ .... (iii)

Multiplying eq. (i) by 2 and subtracting from eq. (ii)

$$\begin{array}{r} 2x_1 + x_2 - 3x_3 = 0 \\ \pm 2x_1 \pm 8x_2 \pm 4x_3 = 0 \\ \hline -7x_2 - 7x_3 = 0 \end{array}$$

$$\Rightarrow \quad x_2 + x_3 = 0$$

Multiplying eq. (ii) by 2 and subtracting from eq. (iii)

$$\begin{array}{r} 3x_1 + 2x_2 - 4x_3 = 0 \\ \pm 4x_1 \pm 2x_2 \mp 6x_3 = 0 \\ \hline -x_1 \qquad 2x_3 = 0 \end{array}$$

$$\Rightarrow \quad x_1 = 2x_3$$

Put $x_3 = t$, then

$x_1 = 2x_3 = 2t$

and $x_2 = -x_3 = -t$

Hence $x_1 = 2t, x_2 = -t, x_3 = t$

Where t is any real number

For $\lambda = -2$:

The given system is

$x_1 + 4x_2 - 2x_3 = 0$ .... (iv)

$2x_1 + x_2 - 3x_3 = 0$ .... (v)

$3x_1 - 2x_2 - 4x_3 = 0$ .... (vi)

Multiplying eq. (iv) by 2 and subtracting from eq. (v)

$$\begin{array}{r} 2x_1 + x_2 - 3x_3 = 0 \\ \pm 2x_1 \pm 8x_2 \mp 4x_3 = 0 \\ \hline -7x_2 + x_3 = 0 \end{array}$$

$$\Rightarrow \quad x_3 = 7x_2$$

Multiplying eq. (iv) by 3 and eq. (v) by 2 and subtracting

$$\begin{array}{r} 3x_1 + 12x_2 - 6x_3 = 0 \\ \pm 4x_1 \pm 2x_2 \mp 6x_3 = 0 \\ \hline -x_1 + 10x_2 \qquad = 0 \end{array}$$

$$x_1 = 10x_2$$

Put $x_2 = t$, then

$x_1 = 10x_2 = 10t$

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and $\quad x_3 = 7x_2 = 7t$

Hence $x_1 = 10t, x_2 = t, x_3 = 7t$

Where t is any real number.

Q6. **Find the value of λ for which the following system does not possess a unique solution. Also solve the system for the value of λ.**

$x_1 + 4x_2 + \lambda x_3 = 2,\ 2x_1 + x_2 - 2x_3 = 11,\ 3x_1 + 2x_2 - 2x_3 = 16.$

Solution:

Let $\quad A = \begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{bmatrix}$

Since the system does not possess a unique solution

$$\therefore \quad |A| = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & 2 \end{vmatrix} = 0$$

Expand by R_1

$$1\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & -2 \\ 3 & -2 \end{vmatrix} + \lambda\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 0$$

$$1(-2+4) - 4(-4+6) + \lambda(4-3) = 0$$

$$1(2) - 4(2) + \lambda(1) = 0$$

$$2 - 8 + \lambda = 0$$

$$-6 + \lambda = 0$$

$$\lambda = 6$$

Put $\lambda = 6$ in given system

$$x_1 + 4x_2 + 6x_3 = 2$$

$$2x_1 + x_2 - 2x_3 = 11$$

$$3x_1 + 2x_2 - 2x_3 = 16$$

$$A_b = \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 2 & 1 & -2 & : & 11 \\ 3 & 2 & -2 & : & 16 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & -7 & -14 & : & 7 \\ 0 & -10 & -20 & : & 10 \end{bmatrix} \text{by } R_2 - 2R_1 \quad R_3 - 3R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & 2 & : & -1 \\ 0 & -10 & -20 & : & 10 \end{bmatrix} \text{by } -\frac{1}{7}R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } R_3 + 10R_2$$

The given system reduced to equivalent system

$$x_1 + 4x_2 + 6x_3 = 2 \quad \text{..... (i)}$$

$$x_1 + 2x_3 = -1 \quad \text{..... (ii)}$$

Put $x_3 = t$ in eq. (ii)

$$x_2 + 2t = -1$$

$$x_2 = -1 - 2t$$

Put $x_2 = -1 - 2t$, $x_3 = t$ in eq. (i)

$$x_1 + 4(-1 - 2t) + 6t = 2$$

$$x_1 - 4 - 8t + 6t = 2$$

$$x_1 - 4 - 2t = 2$$

$$x_1 = 2 + 4 + 2t$$

$$x_1 = 2t + 6$$

Hence

$$x_1 = 6 + 2t$$

$$x_2 = -1 - 2t$$

$$x_3 = t$$

For any real value of t.